THE

MATHEMATICAL GAZETTE

EDITED BY

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THE MATHEMATICAL ASSOCIATION

THE Annual General Meeting of the Mathematical Association was held at King's College, Strand, London, on 4th and 5th January, 1954.

On 4th January, the Business Meeting was held at 11.30 a.m., with the President, Professor T. A. A. Broadbent, in the chair. The Report of the Council for 1953 was adopted, and a financial statement given by the Treasurer was received.

The election of Professor W. V. D. Hodge, Sc.D., F.R.S., as President for 1954 was announced. The existing Vice-Presidents were re-elected, and Professor Broadbent became a Vice-President under Rule 19. The Treasurer, the Secretaries and the Editor of the *Mathematical Gazette* were re-elected. Professor R. L. Goodstein was elected as Librarian and Mr. W. J. Hodgetts as Auditor. The following were elected to serve on the Council: Prof. T. Arnold Brown, Dr. I. W. Busbridge, Mr. W. J. Langford, Miss W. M. Lehfeldt, Dr. E. A. Maxwell, Mr. F. J. Phillips, Mr. M. A. Porter, Mr. A. P. Rollett and Miss K. M. Sowden.

The President spoke of Professor Neville's resignation from the office of Librarian, and expressed the deep appreciation the Association felt for the work done by Professor Neville for the Library during his thirty years' tenure of office. On the nomination of the Council, Professor Neville was elected an Honorary Member of the Association.

At 11.45 a.m. Professor Broadbent gave his Presidential address, " Printer's Ink and the Teacher " ; at 2.15 p.m. a discussion on the recently-published *Report of the Association on Sixth Form Geometry* was opened by Professor Neville and Mr. H. Miller ; at 5 p.m. the Principal of King's College, Mr. P. S. Noble, spoke on " Euclid the Artist ".

On 5th January, the first item, at 10 a.m., was a discussion on " Numerical Analysis " between Mr. J. Kershaw and Dr. C. W. Jones. At 11.30 a.m. Mr. C. T. Daltry spoke on " Teaching through the Flash of Insight ". At 2.15 p.m. a discussion on " Unified Mathematics as a Factor in Education " was opened by Mr. K. R. Imeson, Miss W. A. Cooke, Miss K. M. Sowden and Mr. K. B. Swaine. The last item was a paper entitled " The World around Us " by Dr. E. A. Maxwell.

A Publishers' Exhibition was open during the meeting.

REPORT OF THE COUNCIL FOR THE YEAR 1953

Membership.

During the year 16 junior members and 131 other members joined the Association. The number of members on the 1st November, 1953, was 2,503, this total being made up of 8 honorary members, 235 life members, 2,197 ordinary members and 63 junior members. The figures again show an increase in ordinary membership during the year.

It is with regret that the Council reports the death of the following members: Mr. R. H. Birch (1941), Mr. A. S. Gosset Tanner (1912), Mr. C. H. Hardingham (1913), Dr. G. J. Lidstone (1920), Professor W. M. Roberts (1905), Dr. J. Robertson (1944), Mr. D. Smeltzer (1947), Mr. H. Todd (1947), Instr. Cdr. C. S. Watkins (1948).

Professor W. M. Roberts had been Honorary Auditor since 1927, and Mr. A. S. Gosset Tanner had conducted the Problems Bureau since 1928.

Finance.

Expenditure for the financial year ending 31st October, 1953, amounted to £4,198 12s. 2d., and income to £3,831 3s. 11d.; there was thus an excess of expenditure over income of £367 8s. 3d. The balance in Bank and cash on 1st November, 1953, was £320 15s. 7d.

The cost of printing the Higher Geometry report was £785 3s. 7d., and this very large bill—which had projected its shadow in advance—has been met without reduction of the War Loan reserve. This result has been made possible in great measure by those members—there are now nearly 450 of them—who have signed seven-year covenants. If the estimates for 1953–54 prove to be accurate it will be possible to set aside a further £1,000 for printing reports during the next year, and it is conceivable that the financial structure of the Association may gradually become less like a hasty flood defence and more like a planned irrigation system.

Members who have not already signed a covenant, and are in a position to do so, can greatly help the Association by signing at once. There can be no doubt that the additional income which comes to the Association through this scheme will have a vital effect on the future. Two life members have done their bit by making a donation of a guinea; another life member has generously restarted payment of annual subscriptions and has signed a covenant for seven years. While members show such enthusiasm, and Branches continue to recruit new members, planning for the future will once more become a rational activity.

The Mathematical Gazette.

The number of books received for review continues to increase. There has been no fall in the number of articles and notes submitted for publication, but the proportion of these dealing with early and middle school work remains unduly small.

The Library.

For reasons given in the Council's Report for 1952, the Library has been transferred to University College, Leicester, and is now able to offer normal library service to members.

Professor Neville felt that the interests of the Association could now best be served by a Librarian in close touch with University College, Leicester, and the Council has therefore accepted, with deep regret, his resignation from the post of Honorary Librarian, which he has held since 1923. In this time,

Professor Neville has lavished on our Library the vast bibliographical knowledge and skill of which he is master; as a consequence, the Library has grown from a small and unimportant collection of odd volumes into a valuable store of books and periodicals having special reference to the development and present condition of the teaching of mathematics. Practically nothing has been bought, accessions being almost entirely by gift or exchange; now that the Library has recovered from the disturbances of the war and post-war years, it is to be hoped that the excellent custom of members giving books to their Library, a custom carefully fostered by Professor Neville, will be revived and extended.

The Council wishes to place on record its gratitude to Professor Neville for his long and devoted services to the Library of the Association.

The Teaching Committee.

The Teaching Committee met at Sheffield on April 8th and 11th, 1953, and received reports from the secretaries of its nine sub-committees. The re-draft of the *Report on the Use of Visual Methods in Teaching Mathematics* was approved, and this Report should appear as a 20-page leaflet, with illustrations, early in 1954. The separate draft reports on Mathematics in Primary Schools and in Technical Colleges were discussed at length; considerable modifications were recommended; the sub-committees have been working on them during the summer, and the amended drafts will be laid before the new Teaching Committee on 6th January, 1954.

The remaining sub-committees have continued their work, except the one on the teaching of the history of Mathematics and this will resume early in the new year. The work of the sub-committee on Sixth Form Algebra is especially well advanced.

The Modern Schools sub-committee are sending an outline of their draft report to the Branches, in the hope that discussion may be stimulated and further suggestions from teachers with experience of teaching in Modern Schools sent to the sub-committee.

The Report entitled *The Teaching of Higher Geometry in Schools* was circulated to all members of the Association during the year; so also was a reprint of the report of the 1943–44 Conference on School Certificate Mathematics with its suggested Alternative Syllabus. The *List of Books suitable for School Libraries* has been brought up to date and will be available early in 1954.

The Standing Sub-committee of the Teaching Committee has dealt with various enquiries, including one from the Civil Service Commissioners about a proposed syllabus for Pure-and-Applied Mathematics as a third subject primarily intended for candidates who study mathematics but who do not specialise in it. It has also held a joint meeting with the Standing Committee of Council to discuss a letter received from the Joint Examinations Board (the " Ninth Examining Body ").

It will be a matter of great regret to all members of the Teaching Committee that Mr. M. W. Brown is unable to continue as its Secretary after 1953 on account of the pressure of other work; he has rendered most valuable service during his term of office.

The Branches.

The membership and the work of the Branches continue to flourish. Many interesting meetings have been held, as can be seen from the periodic reports which appear in the *Gazette* insets.

Problem Bureau.

The late Mr. Gosset Tanner continued to deal with applications for solution almost up to the day of his death. Since then, there have been many queries mostly for solutions to Cambridge Scholarship papers. All the members of the late Mr. Gosset Tanner's " staff " have agreed to carry on, and their number has been increased by the addition of Mr. D. A. Young. Thanks are also due to several specialists who deal with occasional questions beyond the ordinary scope of the Bureau's work. Mr. A. J. Gosset-Tanner has been most helpful by sorting and forwarding applications, and making his father's mathematical papers available to the Association. Mr. D. A. Young has extracted the more recent and useful papers from a vast accumulation dating back to the last century.

Officers and Council.

Council wishes to offer, on behalf of all members, its sincere thanks to the President, Professor T. A. A. Broadbent, and to the Officers for the work which they have done for the Association during the year.

GLEANINGS FAR AND NEAR

1755. The forward movement in education is better likened to that of a slow-moving bicycle. The machine, as a whole, creaks forward. Yet any point on the rim of a wheel is moving backward for nearly half the time.—" Who does the Work? ", *Teachers World*, 17th June, 1953, p. 2. [Per Mr C. G. Harrison.]

1756. Some time ago I was asked why water, going down a bowl wastepipe naturally whirled to the right in the northern hemisphere and to the left when south of the equator.

No one apparently had heard of it, but Mr. Stanley Donaldson, of Lamb-lane, Roos, now tells me his theory, and the result of some experiments.

Water falling through a tank outlet, he says, possesses energy, and the solid earth moving round and away from the liquid causes the water to recoil to the right as the earth turns to the left. The water increases its rotary movement as it falls to the centre and centrifugal force keeps the centre of the whirl open, a column of air being drawn through as the water moves out.

However, the water can be made to alter its direction of whirling. Mr. Donaldson experimented with a large plant-pot filled with water from the rainwater tub. On one occasion he noticed it turned to the left, so he tried again, starting to whirl the water to the left with a stick, and it continued in that direction contrary to the natural movement.

He next suspended the plant-pot on string, filled it with water, and found that, as the water runs out, so the plant-pot turns round with it.

This, he thinks, supports a theory he has concerning gravitation, that it is not so much the earth itself that is spinning as the space around that is carrying the earth with it, and that the earth, being in the centre of the pressure, is therefore kept in position, as is everything on it, whereas if the earth itself actually caused the turning, everything on it would have a tendency to fly off. —*Daily Mail*, 26th May, 1953. [Per Miss H. Bromby.]

THE TEACHING OF MECHANICS IN SCHOOL AND UNIVERSITY.

By E. A. Milne.*

Abstract.

The study of mechanics and the solution of problems in mechanics are an intrinsic part of the subject of mathematics as taught in British schools and universities; the subject is associated with such names as those of Routh, Lamb, Loney, Love. It is perhaps a peculiarly British tradition, well worth preserving as a discipline of the first order, comparable with Latin and Euclidean geometry; many examination candidates, even perhaps theoretical physicists of distinction, can write learnedly on atomic physics, and yet find the greatest difficulty in solving dynamical problems requiring a firm grasp of principle *together with facility in applying it*. The present logical unsatisfactoriness of quantum mechanics in certain aspects may be due to want of interest in the foundations of mechanics amongst its exponents.

Three stages may be discerned in the study of mechanics, namely (*a*) at school, (*b*) by undergraduates, (*c*) by researchers. In (*a*) the Newtonian laws, supplemented by such principles as that of the transmissibility of force, may be taken as experimental dogmas; in (*b*) they should be derived from ideal experiments, from which the notion of mass is isolated through the adoption of Newton's Third Law as an axiom, following Mach; in (*c*) the tacit assumptions of the existence of rigid bodies and the "uniform flow" of time should be subjected to criticism. Throughout, an attempt should be made to teach mechanics in terms of vectors directly; the powers of vector analysis in this field have been much developed in recent years by S. Chapman and his followers, and yield beauties and economies of thought undreamed of by the older writers.

I had first better state such claims as I have to discuss the teaching of mechanics at all. First of all, at school, I had a double instruction in mechanics. I was taught by highly competent mathematics masters the mathematical side of mechanics, up to the standard of entrance scholarship at Cambridge; and I was taught by a competent physics master the experimental side of mechanics, up to distinction standard in the Oxford and Cambridge Higher Certificate. I think I did justice to both courses, for I won my entrance scholarship at Trinity in Mathematics and Physics and three years in succession obtained a Distinction in Mechanics in the Higher Certificate. And I think I can still recapture my former schoolboy self's attitude to the two courses; and the schoolboy's attitude is perhaps worth taking into account. My attitude to the course in practical mechanics was that it was jolly interesting, and good fun, that I enjoyed the experiments—Atwood's machine, Fletcher's trolley, experiments with a fly-wheel and a tuning fork, experiments with a pendulum and a tuning fork, experiments on friction, experiments on the bouncing of steel balls on steel plates, experiments on elastic strings; but that these experiments added not one whit to my understanding of the principles of mechanics, or to my power of solving problems

* While clearing up papers left by the late Professor E. A. Milne, F.R.S., who died on 21st September, 1950, I came across the MS of a contribution which he was to have made to a discussion on the teaching of mechanics planned for one of the sessions of the British Association at Dundee, on 2nd September, 1939. Owing to the outbreak of war this session was never held and only an abstract of Milne's remarks has been published so far. (*The Advancement of Science*, **1**, 196, 1940.)

Due to the courtesy of the British Association, it is now possible to publish Milne's projected address in full, subject to such editing as was necessary to make it self-contained. In the main, this consisted in the elimination of references to other speakers in the symposium. Although some of Milne's comments, particularly in the final paragraph, applied with greater force in 1939 than today, the core of the address is of permanent value as a masterly exposition of the point of view of one of the greatest applied mathematicians in our time.—G.J.W.

in mechanics. As to my belief in the validity of the principles of mechanics it was rather impaired than strengthened by these experiments, for the experimental errors were usually distressingly large. In fact, the success of an experiment was judged not by the intensity with which it strengthened my convictions as to the truth of the principle and laws of mechanics, but by the smallness of the experimental errors, i.e., departures from a law previously credited on authority. That is to say, the experiment, by the nature of the resources available, has to be verifications of deductions already made rather than inductions leading to fundamental laws. This is right and proper. It takes the genius of a Hooke, a Huyghens, a Galileo, or a Newton to make deep inductions ; and the inductions that the average schoolboy is likely to make will be more likely to be shallow generalisations of the particular experiment performed rather than fundamental inductions. For example, a schoolboy could derive the laws of angular acceleration of a flywheel under different applied moments and different moments of inertia, but he could hardly get deeper. Each experiment does little more than lead to a separate empirical formula. Theoretical mechanics consists of the attempt to deduce all the separate empirical formulae from a very few general principles, and the fewer the better. Although I would agree that to some boys and girls acquaintance with the fundamental concepts used in mechanics may be made more real by conducting experiments, I can only say that it was not so in my case. The fact is that to understand mechanics is hard work, and any attempt to avoid this hard work—if you like, this tedium and drudgery—by means of kindergarten methods is likely to end in failure. I may mention in passing that this has been my personal experience in other branches of physics—that some practice in the experimental side is of no use whatever to the theoretically minded person ; it hinders rather than advances. This is a thing that experimental physicists find difficulty in understanding. I wish I had at my disposal all the hours I have wasted in unsuccessful experimental work, unremunerative in the better sense. My attitude as a schoolboy to the other and more substantial side of the courses—the mathematical side—was that the principles were mainly always so simple to understand that they hardly made an impression on the mind, whilst examples on the principles were so hard that without guidance one often could not solve a new kind of example at all.

I think that the moral of this is clear. One does not get a grasp of the principles of mechanics by thinking about them, however intensely ; one does not get a grasp of the principles of mechanics by performing mechanical experiments—one merely acquires the outlook of an engineer, a man for whose profession I have an almost envious admiration, but whose profession is not the principles of mechanics. One gets a grasp of the principles of mechanics by carrying out the solutions of mathematical problems in mechanics ; and it is only by whetting one's teeth on problems that one can form the dimmest realisation of how little one understands the principles by merely reading about them.

For such problems to serve their proper purpose, they must be carefully designed. They must be designed with particular principles in mind, and should exemplify these principles explicitly and not lose themselves in the sandy shallows of algebra. Too many of the problems set in current examination papers, or occurring in current textbooks, fail to come up to this level : I often react in disgust from the material thus served up, which exemplifies devices and nothing else. But if one takes the trouble, as I feel no diffidence in proclaiming that I nearly always do, to construct fresh examples when setting an examination paper, genuinely illustrative of principles, one at once finds that so far from mechanics being, as many theoretical physicists contend,

an easier subject than the other branches of applied mathematics, the application of its principles to illustrative problems offers more difficulties to students than do these other branches. It is found, for example, that if one sets a mixed paper containing bookwork and illustrative examples on statics, dynamics, hydrostatics, optics, gravitation and applications, spherical harmonics, candidates who can do quite good work in the later mentioned of these subjects are quite unable to make any showing at, say, the harder problems of rigid dynamics. A candidate for honours who can successfully write out a solution of a problem involving spinning and rolling motion, or motion relative to rotating frames, can be marked up at once as likely to get a First Class ; the Second Class candidates at once pick out the problems in optics and attractions, and leave the rigid dynamics and three-dimensional statics severely alone. The fact is that though the formulation of the principles of mechanics is extremely simple in appearance, their applications are further removed from their immediate expression than in other branches of applied mathematics. Force and work and angular momentum, not to mention angular velocities, inertia tensors and moving vectors, are more difficult things to grasp than a ray or an optical axis ; and when they have been once grasped, there is little additional difficulty in grasping the ideas of a scalar potential, an equipotential surface, a conductor or a magnetic doublet. It is necessary to teach mechanics before these other branches ; but it must be remembered that at the stage at which the respective ideas are first encountered, the fundamental ideas of mechanics are more difficult than, say, those of magnetism or electricity.

Take another context—that of Lagrange's equations. Every examiner in mechanics knows that it is easy to get candidates to *use* Lagrange's equations to solve problems of simple standard types ; and also that it is easy to extract from candidates the simple standard proof of those equations given in books. Every examiner knows too that it is much harder for the candidate accurately to write down the kinetic and potential energies of a system of relatively moving parts than to write down the corresponding Lagrangian coordinates. But it is another matter to be able to say that a candidate *understands* Lagrange's equations. For, if the examiner sets a problem equivalent to the establishing of Lagrange's equations from first principles in a particular case, if, for example, the internal constraints or the external forces, in a given dynamical system, involve the time explicitly, he readily finds that much of what the candidate writes down is parrot language, pumped out without understanding. The fact is, again, that Lagrange's equations are intrinsically very difficult to understand ; they are like a marvellously perfect motor-car, which can be driven without understanding of the concealed complexity ; but the business of the teacher of theoretical mechanics is to ensure not only the use and command of power, but the exercise of understanding.

It is for this reason that I regard mechanics as a discipline of the first order, comparable with Latin and Euclidean geometry. It is possible to criticise Latin and Euclidean geometry and mechanics on the same grounds—namely, that the time taken up by them is out of proportion to their importance. It all depends on what one means by importance. I suppose a schoolboy would have difficulty in saying which he regards as the dullest—mechanics, Latin prose or geometry. Certainly I myself remember finding mechanics duller than pure mathematics, Euclidean geometry duller than differential calculus, Latin prose duller than French prose. But, confining ourselves to school standards, who would suggest replacing these three analytical disciplines by, say, the romantic study of Shakespearean verse or of geography ? Importance must be measured by centrality of position. Every time an atomic physicist refers to the *mass* of an electron, every time he refers

to a momentum or a pair of conjugated coordinates, he is appealing to a nomenclature derived from mechanics, whether he has properly in mind or not all the conditions under which the use of these terms is valid. In my view it is difficult to over-estimate the importance of mechanics owing to this centrality. It is no accident that mechanics is a fine discipline, that dynamics offers the hardest nuts to crack of any of the elementary branches of mathematics. Proficiency in dynamics is evidence of a misspent youth only when the dynamics is confined to the collisions of billiard balls.

So much for my schoolboy experiences of dynamics and my experience as an undergraduate. May I be allowed to say a few words about my experiences as a university teacher? It was my good fortune, when beginning to lecture on dynamics and statics in 1924, to have a conversation with my former teacher Prof. S. Chapman, in which he explained to me his then recent discovery that *vector methods* could be used with success, not merely for the elegant transcription of bookwork, not merely as an abbreviation for triplets of equations, but as an actual practical technique, both in the solution of examples and in research. At first I did not believe him ; I had been brought up, after the manner of the then current textbooks, to regard vectors as elegant toys, tools of precision, if you like, but tools to be kept usually in velvet-lined cases, and certainly not to be taken out if the weather was at all inclement. But I had sufficient confidence in Chapman to try them for myself. I took Lamb's *Higher Mechanics* and systematically solved nearly every example by vector methods. This involved developing a good deal of vector technique not found in textbooks, even those devoted explicitly to vectorial mechanics ; but the result was that many examples, considered by Lamb as hard, became not only easy but an aesthetic pleasure to solve ; and that problems considered by Lamb as beyond the range of elementary dynamics became easily accessible. Since then I have regularly taught mechanics in terms of vectors alone. The result is that the subject becomes transformed. The students have the exhilaration of acquiring a new technique, a new calculus, not for show, but for the most beautiful and most economical solution of examples. A further advantage is that the difficult spatial intuitions required in such unnatural devices as the use of Euler's geometrical relations in the theory of spinning tops became unnecessary. Vector methods always automatically, without mental stress, take full account of such troublesome things as the *signs* of spins, moments about lines and the like. In fact, vector methods bear the same relation to the poor cumbrous methods of the average textbook on dynamics as Cartesian geometry bears to Euclidean ; they provide a direct method of attack. Added to this, they avoid the logical absurdity by which general principles are framed and enumerated in vector form, the results split up into sets of scalar component relations and then recombined to give the eventual results concerning the actual motions concerned.

Another advantage of the vector technique is that it concentrates attention on the essential dynamical aspects of the motion under consideration. A dynamical problem is solved when we can see the cinema pictures of the motion—when we can announce the velocities as functions of the positions. This is achieved by vector methods by one less integration than the corresponding scalar component type of solution, where we laboriously integrate differential equations twice with regard to the time t, to give coordinates as functions of t, then re-differentiate these to give velocities. I have no time to give examples. Let it suffice to say that all questions involving the steady precession of spinning objects—all those harder problems of the rolling and spinning of spheres on spheres free to move, and other combinations of planes, spheres, cylinders and cones—can be treated readily by pure vector methods,

using vectors not as an abbreviation or a code or language, but as the very essence of the dynamical embodiment of the problem.

By reason of these advantages, I think it desirable that university students should be trained in the use of vector methods from the beginning of their course. I have tried this for the last 15 years at Cambridge, Manchester, and Oxford, and invariably worked up a state of intense enthusiasm for the new vector methods. The dullness which, like the proverbial poultice, falls on a course in mechanics of the usual type never arises when vector methods are employed. It should be mentioned that in such a course one should never *apologise* for using vectors ; one apologises only on those rare occasions when owing to the stupidity of the examiner or the writer of the textbook one is driven to solve part of a wholly vector problem in scalars. A considerable advantage of the use of vector methods is that owing to their superior power it is readily possible to find out which of the examples in a given examination paper, or given chapter in Lamb's *Higher Mechanics*, are correct and which incorrect! It should also be mentioned that one should not decide that a question is unsuitable for solution by vectors merely because vectors are not mentioned in its enunciation.

As regards the teaching of the principles of dynamics—I mean the accurate statement and status of them—I have not time here available to go into what I believe to be the true position, namely that laws of dynamics can be shown to be simply logical theorems in which the passage of time has been correctly interwoven. This I shall deal with in the near future in an address to the London Mathematical Society.* This anyhow would be an unsuitable method by which to teach mechanics. Mechanics has probably to be introduced in a purely dogmatic fashion to schoolboys. (This may account for the extreme dogmatism of many atomic physicists, whose grasp of the principles of dynamics is often singularly unsubtle.) But for the university course, some analysis of the origin of the concepts of mass and force must be traced, and in this regard it is difficult to do better than follow Mach. At any rate when one attempts to do the thing for one's self, one usually finds that one has only rediscovered Mach. In Mach's presentation, the equality of action and reaction, Newton's Third Law, is not adopted as an empirical law, but adopted as an axiomatic definition leading to a satisfactory definition of *equal* forces and hence in due course of mass and force, provided of course that we close a Nelsonic eye to the impossibility, without far more advanced considerations, of defining a Galilean frame. This is in accordance with the true development of any science—the use of axioms and postulates not as empirical dogmas but as defining propositions completing the delimitation of the field of discovery ; just as the axiom of parallels is not an empirical fact leading to Euclidean geometry, but a proposition necessary to complete the definition of the Euclidean plane.

A science as it develops passes through a number of definite stages. So does the mind of a growing boy or man. It is important then to suit the stage of development of the science being taught to the stage of development of the student's brain. It would be absurd to teach to schoolboys the last stage—the deductive stage—in a mature science ; just as it would be absurd for university students to be content with the early primitive stages of the science. We may distinguish four stages in the evolution of a developed science :

(1) The collection and classification of facts.
(2) Inductive generalisations and the enumeration of empirical laws.
(3) The introduction of undefined concepts as a basis for the construction of a deductive science.

* *Journal London Math. Soc.*, **15**, 44–80 (1940).

(4) The purely logical stage, wherein all propositions previously held
empirical laws are found to be either defining axioms or deriv
theorems.

Geometry has passed through all these stages. Stage (1) and (2) presumab
occurred in the pre-Greek era. The Greeks gave a deductive basis to geometr
though retaining certain essentially undefined concepts in their treatmen
These were removed in modern axiomatic treatments. In dynamics the co
lection and classification of facts is typified by the work of Kepler. Newto
was the prince of inductive generalisers, of whom other representations we
Galileo and Huyghens. The stage of using undefined concepts is typified b
the work of Einstein, who employs the undefined notions of uniform tim
rigid body, interval, space-time, etc. In our own day dynamics is undergoin
the last transformation, to become a purely deductive science.

The study of mechanics through experiments belongs to the earlier of the
various phases, and may be suitable as an introduction of the subject. In th
later school stages, however, the necessary inductive generalisations must b
made. In the university stage, an attempt may be made to put dynamics on
partially deductive basis in a variety of ways, either starting from Newton
laws, or from conservation laws of energy and momentum—energy an
momentum being taken as indefinable concepts. But a rigorously critic
attitude should be maintained throughout, and the student shown where h
has made tacit assumptions, even where these assumptions cannot, at tha
stage, be removed.

I fear that today mechanics as a subject is threatened. In one importa
English university, a cradle of the teaching of mechanics, the chair of applie
mathematics has been converted into one of theoretical physics, and take
away from the department of mathematics. While I would not for a momen
question the right of a university to its autonomy and to make such change
as it thinks fit, I am sorry for this tendency. As an educational instrument,
finished subject like mechanics is incomparably a finer study than the pr
visional and tentative, unthought-out dogmas of a subject so uncertain
itself as what is called today theoretical physics—a term anyhow I do no
understand. I do not see why the good old Scottish term *Natural Philosoph*
is not wide enough to include any topic of physics that is worth abstract study
It should also be remembered that, *whilst a knowledge of the principles*
mechanics is essential to all science, not all students need to acquire a mi
understanding of atomic physics, a subject extremely unsuitable as a
educational instrument. This discussion will have served a useful purpose i
it strikes a shrewd blow in favour of mechanics as a school and universit
subject.

E. A. M

1757. At the 1927 midwinter auction Fromm pelts did not establish indi
vidual record prices, but the crop as a whole had brought a robust cheque—
$785,153!—Kathrene Pinkerton, digest of "Bright with Silver", *Reader
Digest*, (September 1953), p. 147.

I believe this is the largest number I have met since Professor Littlewood
article in *Gazette* No. 300. The term "robust" is one of extreme modesty
for (unless I err)

$$785{,}153! > 10^{10^{6\cdot 6}},$$

and although mere astronomical numbers are useless for comparison, som
idea of the immensity is given by stating that, expressed in figures in ordinar
notation, it could not be written in a book the size of the Bible (given that th
Bible contains 3,566,480 letters). [Per Mr. C. G. Harrison.]

MATHEMATICS AND THE CHILD. III.

The Use of Mistakes in the Teaching of Mathematics.

By C. Gattegno.

To the mathematician, at least in the field of elementary mathematics, it is immediately apparent what is true and what is false, and for the teacher of mathematics it is the truth, the correct answer, the right reasoning, that are all-important.

In this article, I propose to examine the other side of the picture, to consider the mistakes made by our pupils and to see how they can be turned to advantage in our work as teachers.

There is no disguising the fact that, generally speaking, little or no use is made of the opportunity presented by our pupils' capacity for making mistakes, and yet it is only these mistakes that can throw a light on the significance of our activity as teachers of mathematics. Though obviously we ultimately come to the point at which the mistakes must be corrected and the possibility of their recurrence eliminated, I would suggest that we should do well to curb our tendency to correct, and develop the habit of incorporating into our lessons the observations that we cannot fail to make in marking homework or in using an oral approach with our classes.

It is man's privilege to make mistakes; only through experience, experience that is often painful, does man learn and acquire some degree of wisdom. In the teaching of mathematics, the opportunity for gaining true understanding through experience is too often reduced to the minimum. There is always someone who knows, who can produce the right answer, which is imposed upon those who can not. But how often must the teacher make the same correction, and how many children reach the end of their school career under the impression that only their more fortunate fellows who are "mathematical" can hope to avoid the mistakes which to them are inevitable?

I would suggest that a different approach to this situation is possible. With the help of my students, I made and analysed a collection of mistakes made by children learning mathematics, with a view to discovering what could be learnt from them. It would not be appropriate here to give the list, and every teacher could certainly draw up his own, but I shall indicate a certain number. They are dismal evidence of the defects of the work we are doing, but they can be put to useful purpose if we are prepared to learn from them what they have to teach.

1. Mistakes in elements of algebra:
   - (*a*) failure to understand the meaning of signs,
   - (*b*) incorrect use of brackets, both with and without (*a*),
   - (*c*) incorrect squaring of $a+b$ and $a-b$,
   - (*d*) substitution of ab for $a+b$ and of $2x$ for $x \times x$,
   - (*e*) substitution of division for square root; *e.g.* $x^2=16$, $x=8$,
   - (*f*) substitution of subtraction for division; *e.g.* $3x=15$, $x=12$; $ab=c$, $a=c-b$,
   - (*g*) solution of quadratics with the unknown on both sides of the root

     e.g. $x=\dfrac{-bx \pm \sqrt{b^2x^2-4ac}}{2a}$,

   - (*h*) incorrect simplification of fractions: *e.g.* $\dfrac{x+a}{x+b}=\dfrac{a}{b}$ or $\dfrac{2x+3}{2y+7}=\dfrac{x+3}{y+7}$,
   - (*i*) wrong use of sign $=$; *e.g.* $3x+5=14=14-5=9=\frac{9}{1}$,
   - (*j*) inadequate understanding of rules for operations upon powers; *e.g.* although $x^0=1$, 9^0 is thought to be 9 or 0,
   - (*k*) various combinations of the above.

2. Mistakes in geometry related to :

(*a*) confusion between what is proved and what is to be proved, and between hypothesis and conclusion,
(*b*) lack of understanding of the notion " converse ",
(*c*) confusion of data and theorems,
(*d*) inability to suggest constructions,
(*e*) difficulties in syllogistic reasoning,
(*f*) difficulties in preserving the data throughout the proof,
(*g*) lack of sense of generality,
(*h*) lack of trained intuition,
(*i*) confusion of names and notions,
(*j*) inability to separate purely geometrical from numerical reasoning,
(*k*) inability to use parts of a figure separately and to argue about their relation to the whole,
(*l*) inability to make a mental representation of the situation in question, particularly in three dimensions.

Arithmetic, calculus, trigonometry and mechanics can all provide us with similar lists. Those given above are sufficiently impressive for our purpose here.

All teachers in secondary schools are familiar with such mistakes, but by no means all see in them an opportunity. For the most part, there is merely an attempt to instil right habits by repetition and drill. Too often we console ourselves, and shirk our responsibility in the matter, by deciding that those pupils who persistently make the same mistakes are dull. In my view, such an explanation is entirely unacceptable. Mistakes are mainly due to mishandling of mental situations on the part of the teacher. How otherwise are we to explain the fact that mistakes are so universal and so similar? Does the explanation not lie in the fact that behind the uniformity of the mistakes there is a uniformity of mental structures of which we are still largely unaware?

Why is it, for instance, that when there is confusion between two operations, the confusion is between addition and multiplication, and between division and subtraction? Why is it that in the proof that parallelograms on the same base and between the same parallels are equal in area, the same obstacle is met by the same teacher every year with different groups of pupils, and by other teachers in other schools, and that the difficulty is overcome in a similar way?

To my mind, there is here a clear indication that such mistakes can, if we so choose, constitute a guiding light into our pupils' ways of thinking. If we are prepared to learn from them, our teaching will no longer be that of mathematics, but of adolescents engaged in the process of acquiring mental structures that are akin to mathematical structures.

In physics, it is axiomatic that the same part of space cannot simultaneously be occupied by two objects, and this accords with our everyday experience. But in geometry, when we have two overlapping figures—a situation only possible in so far as the common element is seen as belonging first to one and then to the other of the two figures—a different principle is involved. This simple fact is, however, overlooked, and we fail to see that the resulting lack of understanding of theorems and proofs must inevitably lead to mistakes. We are prepared to concede that mistakes are related to lack of understanding, but we are not prepared to secure understanding so that mistakes may be avoided. Understanding results from integration in existing structures, or from the creation of adequate new structures. It must mean that we stand " over " the problem in question, whereas the presence of mistakes means

that we are in a position of inferiority, and it is the task of the teacher to reverse this situation.

The example given above illustrates the difficulties created by a contradiction between everyday experience and a mathematical situation. For many children, such a contradiction presents an insoluble problem ; they abandon the effort to understand what is involved and accept the view that mathematics is for the gifted few. The wise teacher, however, will help his pupils to overcome the difficulty by showing them that the figure represents a mental image, and that it is possible to *imagine* a situation in which actual figures overlap, only because we focus on parts of the figure first in this way and then in that. The wise teacher will remove the contradiction by making it plain that in actual fact there is no overlapping (in the sense that the same space is occupied by two different objects), but that the figures can be envisaged as if there were. He will thereby create a new awareness, a new structure that goes beyond the empirical experience, a mental experience essential in mathematics. It is this mental structure that intervenes in the child's mathematical thinking, not the idealised object called triangle or circle. Since it is a mental structure, it can be acted upon mentally, and when manipulated mentally it can give rise to other mental situations consistent with the whole of reality, with physics and with everyday experience.

And what of algebraic mistakes, so common and so much more obvious than those in the field of geometry? Here again the remedy is understanding, not repetition and drill, and the role of the teacher is again that of discovering the mental structure that intervenes in the algebraic thinking required.

Few teachers teach addition and subtraction together, and division and multiplication together, as different aspects of the same operation. In my view, it is precisely this fact that accounts for all mistakes in the four operations, apart from numerical mistakes. The fact that in our text-books different chapters are devoted to multiplication and factorisation, etc., shows that the separation is deliberate on the part of most teachers. Such an arrangement is presumably intended to isolate difficulties, whereas in reality it merely created them.

In algebra all operations are reversible. By their very nature, operations constitute pairs. To become aware of the meaning of algebra is to be able to perform both an operation and its inverse with the same ease. Addition and subtraction are two aspects of the same algebraic situation, and are reducible one to the other. It is therefore essential that the pupil who can add algebraically shall at the same time be able to subtract algebraically, thus avoiding mistakes in transferring terms from one side of an algebraic equilibrium to the other. The same holds for multiplication and division.

The mistakes made through confusion of addition with multiplication point to another source of difficulty. In arithmetic, multiplication of integers is repeated addition, while multiplication of fractions is a much more complex algorism. The same word is used in both cases because there are connections between the two processes. In algebra also, multiplication yields addition when some of the *potential* numbers become actual, *e.g.* when ab becomes $2a$, but in general it is an operation defined, like addition, by its own laws, and distinct from addition in that the identity operation is 1 instead of 0. This structural quasi-identity of addition and multiplication in algebra constitutes a stumbling-block for beginners when numbers vanish into letters, and more attention needs to be given, not to drill, but to the actual formation of the abstract operation, its properties being abstracted from arithmetical situations by the use of problems of the " think of a number " type, as a first step to equations.

Algebra is operations upon operations, and can be grasped without diffi-

culty if it is constructed in this way during the first two years of the secondary school. Operations are reversible by definition, and gain their individuality through specialisation, generalisation then being the result of the combination of the new operation with those already existing. All the mistakes quoted are due to lack of awareness of the operations involved, and this lack is in turn due to insufficient emphasis on operations by teachers who have a different conception of what constitutes algebra.

Operations in arithmetic are part of the mental structure of our secondary school pupils, and can be performed, at least in the grammar schools, with some skill. The problem of teaching algebra is, to my mind, that of passing first from operations upon numbers and quantities (the result being numbers and quantities) to awareness, not of the result, but of the operations themselves, and then to the substitution of operations for quantities. The process might be as follows: from $7 \times 4 + 12 = 40$ to $7 \times p + 12 = 40$, $\therefore\ p = \dfrac{40 - 12}{7}$; to $r \times p + 12 = 40$, $\therefore\ p = \dfrac{40 - 12}{r}$ or $r = \dfrac{40 - 12}{p}$; to $r \times p + k = 40$, and finally to $r \times p + k = h$ and its various solutions. The numerical aspect vanishes slowly and is replaced by a set of operations that are regarded as equivalent since they can be reduced one to the other directly either or through their inverse.

$(a + b)^2$ is the square of a sum, and $(a - b)^2$ the square of a difference; each of them is also an operation upon another operation.

The teaching of algebra is the teaching of the dynamics of operations, and this is a self-checking mental activity that requires no other authority than its own smooth functioning. It is for the teacher to create awareness of the mental processes, to bring the structures into existence and let them function according to their own laws. The mistakes which then occur will either be slips, or will be due to insufficient awareness, and will serve to indicate which structures have failed to integrate with previous systems to form new wholes. Once mental structures are present, they impose themselves on the awareness with an inescapable rigour, and determine the framework within which what is true is immediately recognised. Mistakes then no longer have a place; they are spontaneously eliminated through further efforts towards mastery, which is biologically necessary for mental health.

Our responsibility to our pupils demands that we shall not attempt to give norms and right answers, but that we shall use all our skill and imagination to discover the mental structures required for dealing with our mathematical problems, which originated in some mind endowed with those structures.

C. GATTEGNO

1758. Across the ends of the oval the distance (length of a galaxy) is about 150,000 light years—a distance so great that if we were to express it in miles, there would not be sufficient noughts across a column of this journal to give the figures.—J. H. M. Sykes, "Radar reaches out to the stars", *Everybody's Weekly*, 13th September, 1953, p. 14.

The space required to print astronomical numbers is often exaggerated. This article had previously given the number 6,000,000,000,000 (miles in a light year) in less than half a line in the column. [Per Mr. C. G. Harrison.]

1759. The trials of religious leaders in Bulgaria since the Communists came to power have proportionally almost outnumbered those in the Soviet Union. —*The Times*, 19th May, 1953. [Per Professor L. M. Milne-Thomson.]

THE STURMIAN THEORY OF OSCILLATIONS.

By A. G. D. Watson.

1. The Sturmian theory is concerned with the number of zeros that a solution of a linear differential equation of the second order may have in a given interval of the independent variable. As usually presented (*e.g.* Ince (1)) it appears as a very advanced theory, involving elaborate manipulations and reasoning. The purpose of this paper is to simplify the theory by an approach that makes the results intuitive and somewhat reduces the assumptions made.

Our starting point is a pair of differential equations of the first order

$$\dot{x} = -f(t)y, \quad \dot{y} = g(t)x, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

where f and g are continuous and f is positive, throughout a certain interval (t_0, t_1) of the independent variable t. All quantities are real. Any linear differential equation of the second order

$$p_0(t)\ddot{x} + p_1(t)\dot{x} + p_2(t)x = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

can be reduced to the form (1), by taking

$$f = \exp\{-\int(p_1/p_0)\,dt\}, \quad g = (p_2/p_0)\exp\{\int(p_1/p_0)\,dt\}$$

in an interval in which the coefficients are continuous and p_0 does not vanish. (1) is, however, rather more general than (2) and it has an advantage in picturing the results : if x, y are the coordinates of a point in a plane, (1) gives the velocity vector in terms of the position vector and the time.

We need only one advanced result, that the systems of differential equations we are concerned with have solutions that are fixed uniquely by the initial values and can be differentiated with respect to them. This is known to hold, in fact, for any system of equations

$$\dot{x}_r = f_r(x_1, x_2, \ldots x_n, t), \quad r = 1, \ldots n,$$

that satisfies a " Lipschitz condition "

$$|f_r(X_1, X_2, \ldots X_n, t) - f_r(x_1, x_2, \ldots, t)| < \sum_s k_s |X_s - x_s|,$$

$r = 1, \ldots n$, where the k_s are absolute constants. In particular, it holds if the partial differential coefficients $\partial f_r/\partial x_s$ exist and are bounded. Since (1) satisfies a Lipschitz condition, we see that x and y cannot vanish together unless they vanish throughout the interval. Thus, excluding the solution that is identically zero, we can write $x = r\cos\theta$, $y = r\sin\theta$, where r does not vanish and θ can therefore be taken to be continuous. To find the zeros of x we trace the variation of θ, x vanishing only if $\theta \equiv \frac{1}{2}\pi \pmod{\pi}$. From (1) we have

$$r^2\dot{\theta} = x\dot{y} - y\dot{x} = gx^2 + fy^2,$$
$$\dot{\theta} = g\cos^2\theta + f\sin^2\theta, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

since $r \neq 0$. (3) also satisfies a Lipschitz condition.

2. If, in (3), we write $\theta \equiv \frac{1}{2}\pi \pmod{\pi}$, we get $\dot{\theta} = f > 0$. Thus, at a zero θ is always increasing. It follows that between any two successive zeros θ must increase by π. It is not difficult to construct a rigorous proof of this. An immediate result is : *There can be at most one zero in any interval in which* $g < 0$. For in such an interval the solution can cross $\theta \equiv 0 \pmod{\pi}$ in the negative direction only and θ cannot increase by π. Similarly, there can be at most one stationary value (zero of y).

Since two distinct solutions of (3) cannot have the same value of θ for the same value of t, the position vectors of two linearly independent solutions of (1) can never cross ; if they rotate, they must rotate in step. Thus between

any two zeros of one must lie a zero of the other. In any interval the number of zeros can differ by at most one.

If α is the initial value of θ for a particular solution, the theorem quoted says that $\partial\theta/\partial\alpha$ exists; we now see that it is positive.

3. Consider now two distinct systems (1), the different quantities being distinguished by suffixes 1, 2. We assert that: *If $f_1 \geqslant f_2 \geqslant 0$, $g_1 \geqslant g_2$ no solution of the first system can be passed by a solution of the second rotating in the positive direction.* In fact, if $\theta_1 \geqslant \theta_2$ at time t_0, then the same is true at any later time. For if α be the initial value of that solution of the second system that has the same value as the given solution of the first at time t, so that

$$\theta_1(t) = \theta_2(t, \alpha),$$

then

$$\dot{\theta}_1 = \partial\theta_2/\partial t + \dot{\alpha}\partial\theta_2/\partial\alpha. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

But $\dot{\theta}_1 \geqslant \partial\theta_2/\partial t$ and $\partial\theta_2/\partial\alpha > 0$, giving $\dot{\alpha} \geqslant 0$. Hence $\alpha \geqslant \theta_1(t_0)$ for $t \geqslant t_0$ and

$$\theta_1(t) = \theta_2(t, \alpha) \geqslant \theta_2(t, \theta_1(t_0)) \geqslant \theta_2(t). \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

We are now in a position to enunciate the main results.

Theorem 1. (Weak form of Sturm's first comparison theorem).

If, in two systems of equations (1) *$f_1 \geqslant f_2 \geqslant 0$, $g_1 \geqslant g_2$ and the initial angles can be taken as α_1, α_2 where $\frac{1}{2}\pi \geqslant \alpha_1 \geqslant \alpha_2 > -\frac{1}{2}\pi$, then the m'th zero of x_1 is not greater than the m'th zero of x_2.*

For we then have $\theta_1 \geqslant \theta_2$ throughout. It follows that in any interval x_1 has as least as many zeros as x_2. If the condition on the initial angles is omitted we can still say that x_2 can have at most one more zero than x_1.

Theorem 2. (Weak form of Sturm's second comparison theorem.)

If, with the hypotheses of Theorem 1, *x_1 and x_2 have the same number of zeros in the semi-closed interval $t_0 \leqslant t < t_1$, then the residues modulo π of the angles at t_1 satisfy $\frac{1}{2}\pi \geqslant \theta_1 \geqslant \theta_2 > -\frac{1}{2}\pi$.*

This is now obvious. Since, in the interval $\frac{1}{2}\pi > \theta > -\frac{1}{2}\pi$, $\tan\theta$ is an increasing function of θ, we can express the hypotheses and results of these two theorems in terms of y/x instead of θ, taking y/x as $+\infty$ when $x = 0$.

The strong forms of the comparison theorems are obtained by replacing the inequalities by strict inequalities in the hypotheses and results of the weak forms. If $\alpha_1 > \alpha_2$, it is evident that $\theta_1 > \theta_2$ throughout, while if $g_1 > g_2$ we have $\dot{\theta}_1 > \dot{\theta}_2$, and so $\dot{\alpha} > 0$, unless $\theta_1 = \theta_2 \equiv \frac{1}{2}\pi \pmod{\pi}$, which cannot happen over an interval. On the other hand, if $f_1 > f_2$, we can still have $\dot{\theta}_1 = \dot{\theta}_2$ if $g_1 = g_2 = 0$ and $\theta_1 = \theta_2 \equiv 0$. This gives

Theorem 3. (Strong comparison theorems.)

If, with the hypotheses of Theorem 1, *either $\alpha_1 > \alpha_2$, or either of the strict inequalities $f_1 > f_2$, $g_1 > g_2$ holds at the beginning of the interval, then the m'th zero of x_1 is less than the m'th zero of x_2; and if the number of zeros in $t_0 \leqslant t < t_1$ is the same, the residues modulo π of the final angles satisfy $\frac{1}{2}\pi \geqslant \theta_1 > \theta_2 > -\frac{1}{2}\pi$; unless $\alpha_1 = \alpha_2 = 0$ and $g_1 \equiv g_2 \equiv 0$ over an interval from t_0.*

4. Numerical results can be obtained by using as comparison systems those in which f and g are constant. Since these can be solved explicitly we at once have limits for the number of zeros possible in a given interval. Thus we derive

Theorem 4.

If $0 < a \leqslant f \leqslant A$, $b \leqslant g \leqslant B$, with $B \geqslant 0$, and if m and M are integers such that

$$ab \geqslant m^2\pi^2/T^2, \quad AB < M^2\pi^2/T^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

then any solution of (1) *has at least m and at most M zeros in an interval of length T.*

The main application of these theorems is to the study of eigenvalues of systems containing a parameter. But it is also useful to have them available for general purposes. Thus Bessel's equation of general positive or zero order ν can be written in the form (1) with

$$f=1/t, \quad g=t-\nu^2/t. \qquad \text{.....(7)}$$

Thus no function of the type $kJ_\nu(t)+lY_\nu(t)$ can vanish more than once or have more than one stationary value in $0<t<\nu$. Again, a solution of Bessel's equation of order zero which vanishes when $t=100$ vanishes again before $t=103{\cdot}2$. For in the interval $100\leqslant t\leqslant 103{\cdot}2$ we have $f\leqslant 1/103{\cdot}2$, $g\leqslant 100$; and a solution of (1) with $f=1/103{\cdot}2$, $g=100$ that vanishes at $t=100$ vanishes again at $t=100+\pi\sqrt{(1{\cdot}032)}<103{\cdot}2$.

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1760. MACPHERSON'S LAW.

We met a citizen of Edinburgh whom we shall call Macpherson. Macpherson propounded to us what he called " Macpherson's law of economics ". It greatly appealed to us.

Briefly put, Macpherson's law holds that in any economic relationship Macpherson will always come worse off. If Macpherson has a modest meal, the bill will invariably be greater than the bill presented to his fellow-diners who have consumed more than he has. If Macpherson shares a taxi, the driver will, without hesitation, ask him for payment, and, if it comes to the point, Macpherson's taxi will almost certainly break down on the way to catch an important train.—" A Scotsman's Log," *The Scotsman*, February 27, 1952.

Sir,—In " A Scotsman's Log " I read with satisfaction your recent references to evidences of Macpherson's Law, and am pleased to find that acute observers have noticed such matters, but in the interests of accuracy must ask you to appreciate that the examples mentioned comprise but part of a metaphysical theory, which I formulated long ago in the proposition, " Things are against Man ".

Without cataloguing more important cases to illustrate the point, it is necessary to consider only one of the quoted instances to realise from experience that a freely falling collar stud leaps into positions whose inaccessibility increases as the inverse square of the time available for its recovery. Its trajectory transgresses no physical law, but as the resultant situation contains a malicious quality unaccounted for by the pure objectivity of physics, it becomes necessary to seek the causes in other fields.—*The Scotsman*, March 18, 1952.

Sir,—The publicity given in your columns to the recent discovery of Macpherson's Law has invoked a widespread recognition that shows it must have universal validity.

It may interest your readers to know that this principle has also been recognised south of the Border and formulated as the " Law of Pure Cussedness ". This states that when an individual is faced by a combination of circumstances whose outcome has equal chances of being favourable or unfavourable to him, then the principle of pure cussedness intervenes to ensure that the unfavourable outcome is the one that takes place.

A familiar example is to arrive one minute late at a station. Though there are equal chances that the train may leave on time, or a few minutes late, then pure cussedness will send the train off on time.—*The Scotsman*, March 21, 1952. [Per Mr. A. J. Howie.]

SOME THEOREMS ON CONICS.

By E. D. Camier.

1. Some of the properties proved in this essay can hardly have escaped notice, but judging from the lack of reference, they seem to have been forgotten ; others are given for the sake of completeness, and a few are probably new. It is hoped that in any case all are of sufficient novelty to be of interest.

2. The locus of the point from which the tangents to two conics form a harmonic pencil is in general a conic which passes through the eight points of contact of their four common tangents ; it is called the harmonic locus of the conics, and it plays an important role in projective geometry.

Theorem I. If the harmonic locus of two conics is a circle, their centres are isogonal conjugates with respect to each of the four triangles formed by the common tangents.

For, with any one of these triangles ABC as triangle of reference in homogeneous coordinates, the tangential equations of the conics may be taken as

$$\Sigma \equiv \Sigma p\mu\nu = 0 \quad \text{and} \quad \Sigma' \equiv \Sigma p'\mu\nu = 0$$

and the harmonic locus **F** is **(1)***

$$\mathbf{F} \equiv \Sigma pp'x^2 - \Sigma(qr' + q'r)yz = 0. \qquad \text{(i)}$$

Then, assuming trilinear coordinates,† the conditions that **F** should be a circle are

$$c^2qq' + b^2rr' + bc(qr' + q'r) = c^2pp' + a^2rr' + ca(pr' + p'r)$$
$$= b^2pp' + a^2qq' + ab(pq' + p'q)$$

or $(rb + qc)(r'b + q'c) = (ra + pc)(r'a + p'c) = (qa + pb)(q'a + p'b)$ and these are also the conditions that the centres

$$(br + cq,\ ar + cp,\ aq + pb) \quad \text{and} \quad (br' + cq',\ ar' + cp',\ aq' + bp')$$

of Σ and Σ' should be isogonal conjugates in the triangle ABC, which was any one of the four triangles of common tangents.

3. The steps of the argument in § 2 may be reversed, giving the result (important in itself) that if the centres of two conics be isogonally conjugate with respect to a triangle of common tangents, then their harmonic locus is a circle; and considering the fourth common tangent, there are three other triangles circumscribing the conics, and so, by Theorem I their harmonic locus being a circle, the centres of the conics are isogonally conjugate in them also. Moreover, noting that the harmonic locus has as self-conjugate triangle the diagonal (line) triangle of the quadrilateral formed by the tangents, we see that the harmonic locus is the polar circle of this triangle. Thus we may state the converse.

Theorem IA. If two points L, L' are isogonal conjugates with respect to a triangle (T), Σ, Σ' the conics inscribed in (T) which have L, L' as centres, and (C) the fourth common tangent to Σ and Σ', then L, L' are also isogonally conjugate with respect to each of the three other triangles formed by (C) and the sides of (T), and the eight points of contact of Σ and Σ' with (C) and the sides of (T) lie on a circle, the polar circle of the diagonal line triangle of the quadrilateral formed by these four lines.

4. It is natural now to inquire whether such a pair of isogonal conjugates exists for a given quadrilateral.

* Numbers in Clarendon type refer to the references at the end of the paper.

† When trilinears or areals are mentioned, ratios, not actual coordinates, are understood.

Theorem II. Given any quadrilateral, there exists in general one pair of points such that they are isogonal conjugates with respect to each of the four triangles formed by the sides of the quadrilateral, and also possessing the property that the conics having them as centres and touching three of the sides also touch the fourth.

For we know that the locus of the foci of the inscribed conics of a quadrilateral is in general a circular cubic passing through the six intersections of the sides (**2**). Also the centres of these conics lie on the straight line joining the middle points of the diagonals. Since one of the inscribed conics is a parabola, the latter line must meet the cubic on the line at infinity, and so it is parallel to the asymptote of the cubic (**3**). The remaining intersections will be shown to be isogonally conjugate with respect to each of the triangles formed by the sides of the quadrilateral, thus giving the points mentioned.

Taking one of these triangles as triangle of reference, let the fourth side be $\Sigma x/p = 0$. Writing down the condition that the inscribed conic with foci (l, m, n), $(1/l, 1/m, 1/n)$, namely,

$$\Sigma\{l(m^2+n^2) + 2lmn\cos A\}\mu\nu = 0$$

should touch $\Sigma x/p = 0$, we have the equation of the locus of the foci of the inscribed conics of the quadrilateral as

$$\Sigma px(y^2+z^2) + 2xyz\,\Sigma p\cos A = 0.$$

Also the line joining the midpoints of the diagonals is easily found to be $\Sigma a\rho_1 x = 0$, where for brevity we put $\rho_1 = -ap + bq + cr$, with similar expressions for ρ_2 and ρ_3. Now the equation of the cubic can be put in the form

$$(\Sigma ax \,.\, \Sigma a\rho_1 yz + \Sigma a\rho_1 x.\ \ \Sigma ayz)/2abc = 0.$$

Hence, we see that $\Sigma a\rho_1 x = 0$ is parallel to the asymptote, and that it cuts the cubic in two other points which are the intersections of $\Sigma a\rho_1 x = 0$ with its isogonal transform $\Sigma a\rho_1 yz = 0$, that is, these points are isogonal conjugates with respect to the triangle of reference, and they lie on the centre locus. It may be noted that the focus of the inscribed parabola is at the intersection (other than A, B or C) of the conic $\Sigma a\rho_1 yz = 0$ and the circumcircle, and so the isogonal transforms of the join of the midpoints of the diagonals with respect to each of the four triangles concur at the Wallace point of the quadrilateral.

5. In a quadrilateral, each of the triangles formed by taking three of the sides is in perspective with the diagonal line triangle, the axis of perspective being the fourth side, and the centre of perspective the trilinear pole of this side with respect to the corresponding triangle (**4**).

Theorem III. Each centre of perspective is also the pole of the line joining the midpoints of the diagonals with respect to the inscribed Steiner ellipse of the corresponding triangle.

For, in the notation of § 4, (p, q, r) is the pole of $\Sigma a\rho_1 x = 0$ with respect to $\Sigma a\mu\nu = 0$.

Theorem IV. If the conics of Theorem II touch the sides of one of the triangles ABC of the quadrilateral at R, S, T and R', S', T', the corresponding centre of perspective, that is, the trilinear pole of the fourth side with respect to ABC, is the intersection of the axes of perspective of the pairs of triangles ABC, RST and ABC, $R'S'T'$. By § 4, we may take the centres of these conics to be (l, m, n) and $(1/l, 1/m, 1/n)$, and their tangential equations will then be $\Sigma a\rho_1\mu\nu = 0$, $\Sigma a\bar{\rho}_1\mu\nu = 0$, using the notation $\rho_1 = -al + bm + cn$ and

$$\bar{\rho}_1 = -a/l + b/m + c/n, \quad \text{etc.}$$

The fourth side of the quadrilateral is then

$$\Sigma x/(b\phi_3 + c\phi_2) = 0,$$

where $\phi_1 = m/n - n/m$, etc. The axes of perspective of the pairs of triangles ABC, RST and ABC, $R'S'T'$ are $\Sigma a\rho_1 x = 0$, $\Sigma a\bar{\rho}_1 x = 0$ respectively, and these intersect in the point

$$(b\phi_3 + c\phi_2,\ c\phi_1 + a\phi_3,\ a\phi_2 + b\phi_1)$$

which is the trilinear pole of the fourth side.

6. Returning to the case of a fixed triangle ABC and a pair of variable points L, L' isogonally conjugate with respect to it, and using the notation described in Theorem IA, we consider the radical axis of the harmonic locus circle **K** and the circle ABC.

As in § 5, let (l, m, n) and $(1/l, 1/m, 1/n)$ be the trilinear coordinates of L, L' referred to the triangle ABC. Then in equation (i) of § 2 we may substitute $a\rho_1$, $b\rho_2$, $c\rho_3$ for p, q, r and $a\bar{\rho}_1$, $b\bar{\rho}_2$, $c\bar{\rho}_3$ for p', q', r' to obtain the equation of **K**; it is

$$\Sigma a\rho_1\bar{\rho}_1 x^2 - \Sigma bc(\rho_2\rho_3 + \rho_3\bar{\rho}_2)yz = 0.$$

Since $(\rho_2 + \rho_3)(\bar{\rho}_2 + \rho_3) = 2al \,.\, 2a/l = 4a^2$, we have

$$\rho_2\bar{\rho}_3 + \rho_3\bar{\rho}_2 = 4a^2 - (\rho_2\bar{\rho}_2 + \rho_3\bar{\rho}_3)$$

so that the equation of **K** is easily seen to be also

$$\Sigma ax \,.\, \Sigma a\rho_1\bar{\rho}_1 x - 4abc\ \Sigma ayz = 0.$$

The radical axis of **K** and ABC is therefore $\Sigma a\rho_1\bar{\rho}_1 x = 0$.

Theorem V. If one of the isogonally conjugate points lies on the inscribed Steiner ellipse, then the inconic with centre at the other touches the radical axis of **K** and the circle ABC.

For (l, m, n) lies on the inscribed Steiner ellipse if

$$\Sigma a^2 l^2 - 2\Sigma bcmn = 0$$

which may be written $\Sigma\rho_2\rho_3 = 0$ ‖ and $\Sigma a\rho_1\bar{\rho}_1 x = 0$ touches Σ' or $\Sigma a\bar{\rho}_1\mu\nu = 0$ if

$$\Sigma abc\bar{\rho}_1\rho_2\bar{\rho}_2\rho_3\bar{\rho}_3 = abc\bar{\rho}_1\bar{\rho}_2\bar{\rho}_3\Sigma\rho_2\rho_3 = 0$$

so that the result is evident.

For the proof of the next theorem, the following lemma is required: the conic of which the line equation in general homogeneous coordinates is

$$\Sigma(m/n - n/m)\mu\nu = 0$$

touches the lines $\Sigma lx = 0$, $\Sigma x/l = 0$, the chord of contact being

$$\Sigma(m/n + n/m)x = 0.$$

Denoting $(m/n + n/m)$ and similar expressions by θ_1, θ_2, θ_3, it is easy to see that $\phi_2\phi_3 = 2\theta_1 - \theta_2\theta_3$; then if we use this in the point equation of the conic, the result is obvious, for

$$\begin{aligned}\Sigma\phi_1^2 x^2 - 2\Sigma\phi_2\phi_3 yz &= \Sigma(\theta_1^2 - 4)x^2 - 2\Sigma(2\theta_1 - \theta_2\theta_3)yz \\ &= (\Sigma\theta_1 x)^2 - 4(\Sigma x^2 + \Sigma\theta_1 yz) \\ &= (\Sigma\theta_1 x)^2 - 4\Sigma lx \,.\, \Sigma x/l.\end{aligned}$$

Theorem VI. The conic inscribed in the triangle ABC, with centre at the trilinear pole of the fourth common tangent of Σ and Σ' touches the trilinear polars of L and L', and the chord of contact is parallel to the radical axis of **K** and the circle ABC.

For the trilinear pole of the fourth common tangent is (§ 5)

$$(b\phi_3 + c\phi_2,\ c\phi_1 + a\phi_3,\ a\phi_2 + b\phi_1) \equiv (P, Q, R), \text{ say,}$$

and so the inscribed conic having this as centre is

$$\Sigma a(-aP+bQ+cR)\mu\nu=0,$$

which reduces to $\Sigma\phi_1\mu\nu=0$. By the lemma this touches $\Sigma lx=0$ and $\Sigma x/l=0$ with chord of contact $\Sigma\theta_1 x=0$. But the equation of the radical axis of $\mathbf{K}$ and the circle ABC, namely, $\Sigma a\rho_1\rho_1 x=0$ may be written in the form

$$(\Sigma a^2-\Sigma bc\theta_1)(\Sigma ax)+2abc\,\Sigma\theta_1 x=0,$$

showing that it is parallel to the chord of contact $\Sigma\theta_1 x=0$.

If the points L, L' are such that the radical axis is the line $\Sigma\theta_1 x=0$ itself, we see that they must lie on the cubic

$$\Sigma bcx(y^2+z^2)-xyz\,\Sigma a^2=0.$$

This equation may be written

$$\Sigma ayz\,.\,\Sigma ax-2xyz\,\Sigma a^2=0,$$

from which it is evident that the cubic has triple contact with the circle ABC at A, B and C, and that BC, CA and AB are its asymptotes.

We may summarise a few of its properties as follows: a pair of isogonally conjugate points L, L' in ABC lie on the cubic if

(i) they are also conjugate with respect to the inscribed Steiner ellipse of ABC;

(ii) the inconic of ABC with either L or L' as centre touches the trilinear polar of the other;

(iii) the conic through the six points in which AL, BM, CN, AL', BM', CN' meet the opposite sides of ABC passes through the symmedian point;

(iv) they are also conjugate with respect to the circle whose centre is the orthocentre of ABC, and which passes through the intersections of the nine-points circle and the perpendicular on the Euler line from the symmedian point.

The properties (i) and (ii) follow easily if we note that the equation

$$\Sigma bc\theta_1-\Sigma a^2=0$$

can be put in the forms $\Sigma al\overline{p}_1=0$, $\Sigma(a/l)\rho_1=0$: and in (iii), (iv) the conic and circle are respectively

$$\Sigma x^2-\Sigma\theta_1 yz=0,\quad(\Sigma a^2)\Sigma x^2\sin 2A-2\Pi\sin A\,.\,(\Sigma ax)^2=0.$$

In the present connection the cubic has another interesting property which may be given as follows:

Theorem VII. When L, L' lie on the cubic

$$\Sigma bcx(y^2+z^2)-xyz\,\Sigma a^2=0,$$

the circle $\mathbf{K}$ touches the circumcircles of the diagonal line triangles of the quadrilaterals formed by the sides of the triangle ABC and each of the trilinear polars of L and L', the points of contact being the intersections of $\mathbf{K}$ and the circle ABC.

We first remark that the triangles RST, $R'S'T'$ are respectively homothetic to the diagonal line triangles XYZ, $X'Y'Z'$ of these quadrilaterals. For the equation $-a\rho_1 x+b\rho_2 y+c\rho_3 z=0$ of RS may be put into the form

$$\rho_1\Sigma ax-2bc(ny+mz)=0$$

which shows that RS is parallel to $ny+mz=0$ and this is the side YZ through A of the diagonal line triangle of the quadrilateral formed by ABC and

$$\Sigma x/l=0.$$

The centre of similarity is found as the intersection of the join of $X\ (-l, m, n)$ and $R\ (0, c\rho_3, b\rho_2)$ with the join of $Y\ (l, -m, n)$ and $S\ (c\rho_3, 0, a\rho_1)$; it is $I\ (l/\rho_1, m/\rho_2, n/\rho_3)$. Similarly, the corresponding point I' for $X'Y'Z'$, $R'S'T'$ is $(1/l\bar{\rho}_1, 1/m\bar{\rho}_2, 1/n\bar{\rho}_3)$.

The identity of the relations

$$\Sigma bc\, l(m^2+n^2) - lmn\,\Sigma a^2 = 0, \quad \Sigma al\bar{\rho}_1 = 0, \quad \Sigma(a/l)\rho_1 = 0$$

has already been pointed out.

Now, if I lies on the circle ABC or $\Sigma ayz = 0$, so does I', for the conditions are $\Sigma(a/l)\rho_1 = 0$ and $\Sigma al\bar{\rho}_1 = 0$, which are the same. But these are also the conditions that I, I' should lie on the radical axis $\Sigma a\rho_1\rho_1 x = 0$ of $\mathbf{K}$ and the circle ABC. Hence I, I' are the intersections of $\mathbf{K}$ and ABC when L, L' lie on the cubic.

Also, since the centre of similarity I of the triangles XYZ, RST lies on the circle RST or $\mathbf{K}$, this circle touches the circle XYZ, the point of contact being I. Likewise the circles $\mathbf{K}$ and $X'Y'Z'$ touch at I'.

7. Continuing with the case of a fixed triangle ABC and a pair of variable points L, L' isogonally conjugate to it, we have :

Theorem VIII. With the notation of Theorem IA, if the line LL' passes through a fixed point, the envelope of the fourth common tangent of Σ and Σ' is a conic inscribed in ABC, having the fixed point as centre.

For, if LL' or $\Sigma\phi_1 x = 0$ (in the notation of § 5) passes through (p, q, r), then its pole with respect to the inscribed Steiner ellipse of ABC lies on a fixed line

$$\Sigma a(-ap+bq+cr)x = 0.$$

Hence the trilinear polar of this pole envelopes the inscribed conic

$$\Sigma a(-ap+bq+cr)/\lambda = 0,$$

and this obviously has its centre at (p, q, r).

The points L, L' obviously lie on a cubic through ABC. When the fixed point is the symmedian point, the locus is the cubic curve $\Sigma ax(y^2-z^2) = 0$ which has been studied extensively.

Theorem IX. If LL' passes through the symmedian point of ABC the harmonic locus (circle) of Σ and Σ' touches the nine-points circle.

For the condition that LL' goes through the symmedian point (a, b, c) is

$$\Sigma a\phi_1 = 0 \quad \text{or} \quad \Sigma(b\phi_3 + c\phi_2)\cos A = 0,$$

so that the conic $\Sigma(b\phi_3 + c\phi_2)yz = 0$ is a rectangular hyperbola. But this conic circumscribes the diagonal line triangle of the quadrilateral formed by the sides of ABC and the fourth common tangent of Σ, Σ', touching its sides at A, B and C; and the harmonic locus of Σ and Σ' is the polar circle of this diagonal triangle.

By a theorem due to R. A. Roberts, the polar circle of a circumscribed triangle of a rectangular hyperbola touches the nine-points circle of the triangle formed by the points of contact, at the centre of the hyperbola (5). Hence the circle through the points of contact of the conics Σ, Σ' with the sides of ABC touches the nine-points circle.

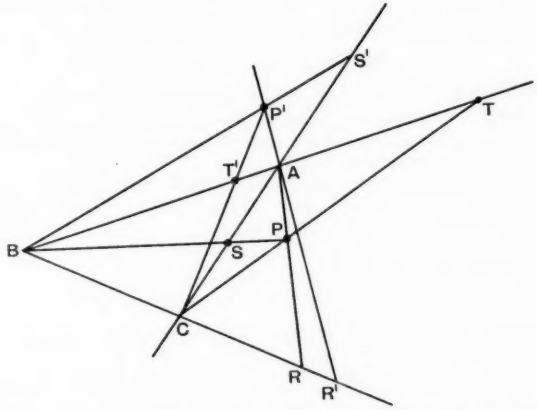
It may be remarked that the points of concurrence P of AR, BS, CT and P' of AR', BS', CT' lie on the rectangular hyperbola mentioned above. For P is $(1/a\rho_1, 1/b\rho_2, 1/c\rho_3)$ and P' is $(1/a\bar{\rho}_1, 1/b\rho_2, 1/c\rho_3)$; and the join of the points $(a\rho_1, b\rho_2, c\rho_3)$, $(a\rho_1, b\rho_2, c\bar{\rho}_3)$ is $\Sigma(b\phi_3 + c\phi_2)x = 0$; these points are the isogonal conjugates of P, P' which therefore lie on the conic

$$\Sigma(b\phi_3 + c\phi_2)yz = 0.$$

8. When one of the conics, Σ, is a parabola, it is clear that the centre of Σ' is the focus of Σ. Hence we have :

Theorem X. If a conic Σ' be drawn with its centre at the focus of a parabola Σ, to touch three tangents to Σ, the six points of contact of Σ and Σ' with the tangents lie on a circle.

We conclude with some properties of this configuration, using the same notation as in the previous paragraphs. The circle $RSTR'S'T'$ will be denoted by **K**.



Theorem XI. The sum of the powers of A, B, C with respect to the circle **K** is equal to the sum of the squares on the sides of the triangle ABC.

For, let $\Sigma qr\mu\nu = 0$ be the tangential equation of the parabola Σ referred to the triangle ABC, the coordinates being areals. Then p, q, r satisfy the relation

$$\Sigma qr = 0. \quad \text{.....(i)}$$

The points R, S, T are $(0, q, r)$, $(p, 0, r)$, $(p, q, 0)$ respectively. Then, assuming $\Sigma lx \,.\, \Sigma x - \Sigma a^2yz = 0$ as the equation of **K**, and substituting the coordinates of R and using (i) we have

$$mq + nr = a^2qr/(q + r) = -a^2p.$$

With the two other relations formed in the same way, we get

$$l = (a^2p - b^2q - c^2r)/2p, \quad \text{.....(ii)}$$

with corresponding values for m and n. Now l, m and n are the powers of A, B and C with respect to **K** and so by (ii)

$$\begin{aligned} \Sigma l &= \Sigma\{-(\Sigma a^2p)qr + 2a^2pqr\}/2pqr \\ &= -(\Sigma a^2p)(\Sigma qr)/2pqr + \Sigma a^2 \\ &= \Sigma a^2, \quad \text{by (i).} \end{aligned}$$

Theorem XII. The intersection P' of AR', BS', CT' lies on the radical axis of **K** and the circle ABC.

For $AS : SC = r : p$, so that $AS = br/(r + p) = -bq/p$, by (i). But

$$AS \,.\, AS' = l,$$

giving

$$AS' = (-a^2p + b^2q + c^2r)/2bq.$$

Similarly $CS = -bq/r$ and $CS' = (a^2p + b^2q - c^2r)/2bq$. Hence

$$CS' : S'A = (a^2p + b^2q - c^2r) : (-a^2p + b^2q + c^2r)$$

and the equation of BS' is

$$(-a^2p + b^2q + c^2r)x - (a^2p + b^2q - c^2r)z = 0.$$

The equations of AR' and CT' are easily written down, and we see that their point of concurrence P' is

$$\{(-a^2p + b^2q + c^2r)^{-1}, \quad (a^2p - b^2q + c^2r)^{-1}, \quad (a^2p + b^2q - c^2r)^{-1}\}.$$

Now the equation of the radical axis of the circles **K** and ABC is, by (ii),

$$\Sigma qr(-a^2p + b^2q + c^2r)x = 0,$$

and it passes through the point P', as a consequence of (i).

Theorem XIII. The trilinear polar of P' with respect to ABC envelopes a fixed conic which is the locus of the centres of conics inscribed in ABC and passing through its symmedian point.

For, if the inscribed conic $\Sigma\sqrt{(Fx)} = 0$ passes through the symmedian point (a^2, b^2, c^2) of ABC, $\Sigma\sqrt{(Fa^2)} = 0$. But if (x', y', z') is the centre of this conic, we have easily

$$2F = -x' + y' + z', \text{ etc.}$$

Hence the locus of the centres is the conic

$$\Sigma\sqrt{\{a^2(-x + y + z)\}} = 0, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)}$$

a conic inscribed in the medial triangle of ABC (**6**). But the condition that the trilinear polar of P',

$$\Sigma(-a^2p + b^2q + c^2r)x = 0 \quad \text{or} \quad \Sigma a^2p(-x + y + z) = 0$$

should touch the conic (iii) is $\Sigma a^2 \cdot b^2q \cdot c^2r = 0$ which is clearly satisfied since $\Sigma qr = 0$.

It is known that if parallels are drawn to the sides of the triangle RST through A, B, C forming a triangle XYZ, then the centre of similarity of XYZ and RST is the centroid G.

Theorem XIV. If the parallels through A, B, C to $S'T'$, $T'R'$, $R'S'$ form a triangle $X'Y'Z'$, then $X'Y'Z'$ is in perspective with RST, the centre of perspective lying on the conic through ABC which has its centre at the symmedian point, and the axis of perspective passing through a fixed point on the Euler line. (It may be remarked that the pairs of triangles XYZ, $R'S'T'$ and $X'Y'Z'$, RST are in perspective in the general case where Σ is not a parabola.)

From the coordinates of P' already found, the equation of $T'S'$ is clearly

$$-(-a^2p + b^2q + c^2r)x + (a^2p - b^2q + c^2r)y + (a^2p + b^2q - c^2r)z = 0$$

and the parallel $Y'Z'$ to this through A is easily found to be $c^2ry + b^2qz = 0$. Thus $Z'X'$ is $a^2pz + c^2rx = 0$ and $X'Y'$ is $b^2qx + a^2py = 0$, so that X' is

$$(-a^2p, b^2q, c^2r).$$

Now we may find the equations of $X'R$ and $Y'S$, namely,

$$(b^2 - c^2)qrx + a^2pry - a^2pqz = 0$$

and

$$-b^2qrx + (c^2 - a^2)pry + b^2pqz = 0.$$

These intersect at the point $(ap \cos A, bq \cos B, cr \cos C)$ and the symmetry of this shows that $X'Y'Z'$ and RST are in perspective.

From the relation $\Sigma qr = 0$, the locus of the centre of perspective is

$$\Sigma a \cos A \cdot yz = 0,$$

a circumconic of ABC with centre at (a^2, b^2, c^2).

For the axis of perspective, we find the intersections of $Y'Z'$, ST and $Z'X'$, TR to be respectively

$$\{p(b^2-c^2),\ b^2q,\ -c^2r\} \quad \text{and} \quad \{-a^2p,\ (c^2-a^2)q,\ c^2r\},$$

and the equation of the line joining them is

$$\Sigma bc \cos A\ qrx = 0 \quad \text{or} \quad \Sigma xqr \cos A/a = 0.$$

Using (i) we see that this passes through the fixed point (tan A, tan B, tan C) on the Euler line, and so the theorem is proved.

REFERENCES.

1. Sommerville, *Analytical conics*, (3rd ed.) p. 283.
2. Loney, *Coordinate Geometry, II*, p. 83.
3. Loney. Cf. Ex. 20, p. 103.
4. Casey, *Analytical geometry* (1893), p. 72.
5. Casey, p. 283, Ex. 44.
6. Casey, p. 339, Ex. 6, or Salmon, *Conics*, (6th ed., 1879), p. 267, Ex. 2.

E. D. C.

1761. Schoolboys sometimes wonder how often they will be able to make use, in later life, of the fact that the sum of opposing angles of a quadrilateral is 180 degrees.—*Southampton Daily Echo*, October 10, 1951. [Per Mr. A. R. Pargeter.]

1762. The reader of this chapter (on Simple Differential-Equation Systems—the last in the book) should have at least a rudimentary knowledge of the differential calculus. Specifically he should know the meaning and notation of the derivative, the second derivative, etc., and should know that the derivative of ax^re^{xb} with respect to x is given by $rax^{r-1}e^{xb}+bax^re^{xb}$ and in particular that $dae^{mx}/dx = ame^{mx}$, where the definition of e is restated below to remind those who may have forgotten. Should there be any foolhardy souls who venture here without this information they may perhaps find it helpful to think of the derivative in terms of the marginal analysis. Thus if $P=f(L)$, where L is the quantity of labour employed by a firm and P is the output of the firm, then dP/dL (the derivative of production with respect to quantity of labour employed) is the marginal product of labour. It should also be noted that one often writes y' for dy/dt, y'' for d^2y/dt^2, etc.—Baumol, *Economic dynamics* (1951), p. 218. [Per Mr. J. Maclean.]

1763. " We dream our dreams and build our airy castles—and often enough that is all—but sometimes something occurs which changes phantasy into reality. The pure mathematician, Pacey, loses all his cherished purity and becomes an applied mathematician."

" No, no," protested Pacey, " no two habits of mind could be more alien to one another than those of the pure and the applied mathematician."—J. C. Masterman, *To Teach the Senators Wisdom, or, An Oxford Guide-Book*, p. 35. [Per Miss M. Pick.]

1764. . . . that is how I feel about Bach. . . . Is he, perhaps, more than anything, a great churchman and a supreme mathematician?—Victor Gollancz, *My dear Timothy*, (1952), p. 169. [Per Dr. B. H. Neumann.]

THE INVARIANT CIRCLES OF A BILINEAR TRANSFORMATION.

By M. P. Drazin.

We say that a bilinear transformation $w=f(z)$ leaves a given locus C in the complex plane " invariant " if C coincides, as a point set, with its image $f(C)$ under the transformation.

It is an elementary fact that the general bilinear transformation leaving a *given circle* $|z-\zeta|=r$ invariant is of the form

$$w=\zeta+e^{i\alpha}\frac{z-\zeta-r\beta}{\bar{\beta}(z-\zeta)-r} \quad (\alpha \text{ real}, |\beta|\neq 1);$$

in this investigation we shall first obtain a " canonical form " applying to *any* bilinear transformation, and then use this to determine those circles (if any) which are left invariant by a *given transformation*, say

$$w=f(z)=(az+b)/(cz+d), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

where a, b, c, d are any given complex numbers (with c, d not both zero).

We may obviously exclude from the outset the case in which $f(z)$ is identically constant, *i.e.* we may suppose that $ad-bc\neq 0$. We shall also suppose, for the present, that $c\neq 0$. Then

$$dw/dz=(ad-bc)/(cz+d)^2,$$

so that $|dw/dz|=1$ if, and only if,

$$|cz+d|=|ad-bc|^{1/2}; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

the locus (2) is a proper circle, known as the *isometric circle* of the transformation $w=f(z)$.

If we make the change of origin, scale, and orientation of axes corresponding to the change of variables

$$\left.\begin{aligned} cz+d&=(ad-bc)^{1/2}z' \\ cw+d&=(ad-bc)^{1/2}w' \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

(with the same determination of the ambiguous sign in each case), we find

$$\begin{aligned} cw&=c(az+b)/(cz+d)=\{a(cz+d)-(ad-bc)\}/(cz+d) \\ &=a-\frac{\sqrt{(ad-bc)}}{z'}, \end{aligned}$$

and therefore

$$w'=(ad-bc)^{-1/2}(cw+d)=(ad-bc)^{-1/2}(a+d)-1/z'.$$

If we define

$$\gamma=\tfrac{1}{2}(a+d)(ad-bc)^{-1/2},$$

then we have

$$w'=2\gamma-1/z',$$

and the relations (3) set up an obvious (1, 1) correspondence between the invariant circles (if any) of *this* transformation and those of $w=f(z)$. It follows that we may, without loss of generality, restrict ourselves to determining the invariant circles of transformations of the special type

$$w=g(z)=2\gamma-1/z,$$

which may be regarded as the result of an inversion in the circle $|z|=1$, followed successively by a reflection in the imaginary axis and a translation of amount 2γ.*

In conformity with standard usage, we shall use the term " circle " to mean *either* a circle of finite *non-zero* radius (a proper circle) *or* a straight line.

* It will be observed that $4\gamma^2=(a+d)^2/(ad-bc)$ is an invariant of the transformation (1) under *any* simultaneous *bilinear* transformation of the z- and w-planes.

Let C be any given proper circle, which we may suppose to be defined by an equation of the form $|z-\rho e^{i\varphi}|=r$, where ρ, φ, r are real, and $\rho \geqslant 0$, $r>0$.

It is, in view of the special nature of the transformation $g(z)$, easy to see that the image $g(C)$ of the circle C will have as a *diameter* the linear segment joining the two points

$$p=g((\rho+r)e^{i\varphi})=2\gamma-\frac{1}{(\rho+r)e^{i\varphi}}, \quad q=g((\rho-r)e^{i\varphi})=2\gamma-\frac{1}{(\rho-r)e^{i\varphi}}$$

(the transforms of the two intersections of C with its diameter through the origin), so that we shall have $g(C)=C$ if, and only if,

$$|p-q|=2r \quad \textit{and} \quad \tfrac{1}{2}(p+q)=\rho e^{i\varphi},$$

i.e.

$$\rho^2-r^2=\pm 1 \quad \textit{and} \quad 2\gamma=\rho e^{i\varphi}+\frac{\rho e^{-i\varphi}}{\rho^2-r^2};^*$$

in other words, C will be an invariant circle of the transformation $w=g(z)$ if, and only if,

either $\quad r^2=\rho^2+1, \quad \gamma=\rho i \sin\varphi,$

or $\quad r^2=\rho^2-1, \quad \gamma=\rho\cos\varphi.$

Since C was a general proper circle, it follows that a necessary and sufficient condition that the transformation $w=g(z)$ have *an* invariant circle is that γ be either pure imaginary or real, *i.e. that* γ^2 *be real.*

Case (i); $\gamma^2 \leqslant 0$: Here, on writing $\gamma=it$, $z=x+iy$ (where t, x, y are real), the general invariant circle

$$(x-\rho\cos\varphi)^2+(y-\rho\sin\varphi)^2=r^2$$

takes the form

$$(x \pm \sqrt{(r^2-1-t^2)})^2+(y-t)^2=r^2 \quad (\text{for } r^2 \geqslant 1+t^2=1-\gamma^2),$$

giving a coaxal system of circles, intersecting in the points $(0, t\pm\sqrt{(1+t^2)})$, *i.e.* in the *fixed points* α_1, $\alpha_2=\gamma\pm\sqrt{(\gamma^2-1)}$ of the transformation $w=g(z)$.

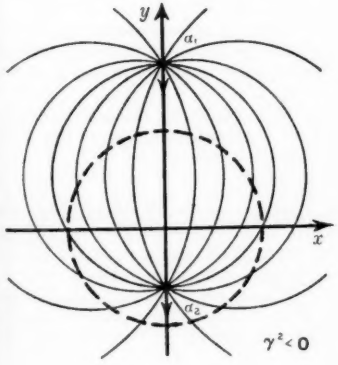


FIG. 1

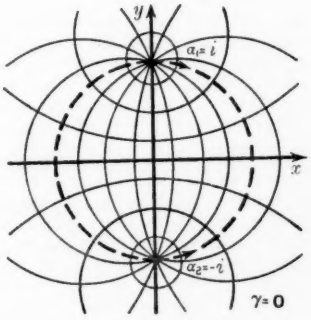


FIG. 2

We have tacitly assumed in this paragraph that $\rho^2 \neq r^2$: however, it is easy to see that C cannot be invariant under $g(z)$ if it passes through the origin.

Case (ii) ; $\gamma^2 \geqslant 0$: Here we have the invariant system

$$(x-\gamma)^2+(y \pm \sqrt{\{r^2+1-\gamma^2\}})^2=r^2 \quad (\text{for } r^2 \geqslant \gamma^2-1).^*$$

In the subcase $0 \leqslant \gamma^2 \leqslant 1$, this system includes the " point-circles " at

$$(\gamma, \pm\sqrt{\{1-\gamma^2\}}).$$

If, however, $\gamma^2 \geqslant 1$, then every circle of the system passes through the points $(\gamma \pm \sqrt{\{\gamma^2-1\}}, 0)$, and we again have a system intersecting in the fixed points. Thus, in both of these subcases, the invariant circles of the transformation $w=g(z)$ form a coaxal system with " base points " at the roots α_1, α_2 of the equation $g(z)=z$, and these base points are limiting points (*i.e.* point-circles) of the system if $\gamma^2 \leqslant 1$, while they are *real* common points (in both senses) if $\gamma^2 \geqslant 1$.

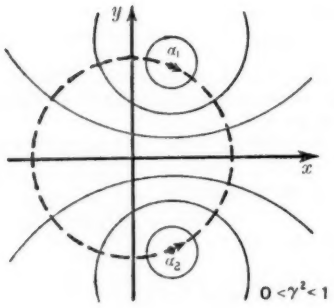


FIG. 3

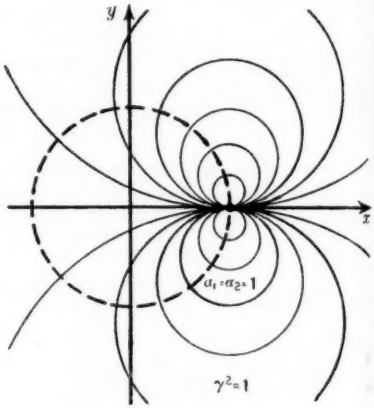


FIG. 4

On turning next to the limiting case $r=\infty$, it is a simple matter to verify that the transformation $g(z)$ can have no invariant *line* unless γ^2 is real, in which case we get the linear element of the appropriate coaxal system (*i.e.* the imaginary axis if $\gamma^2 \leqslant 0$, the real axis if $\gamma^2 \geqslant 0$).

Thus we have now shown that the general (non-constant) bilinear transformation with $c \neq 0$ is equivalent to a " standard " transformation of the type $w=2\gamma-1/z$, which can have an invariant " circle " only if

$$\gamma^2=(a+d)^2/4\,(ad-bc)$$

is *real*, and, when this condition is satisfied, there is a *coaxal system* of invariant circles, and no others (except in the case $\gamma=0$, where there are two orthogonal systems). Further, this coaxal system has for its base points the fixed points of the transformation (which are, roughly speaking, limiting points or common points according as $0<\gamma^2<1$ or not). The configurations of invariant circles for typical values of γ^2 are shown in Figs. 1–5 (of which Fig. 1 is drawn for the case $i\gamma<0$, while $\gamma>0$ in Figs. 3–5) ; the unit circle $|z|=1$ is shown " broken " in each case.

We have now to show that the case $c=0$, previously excluded, can also be

* We may observe, in passing, that the relation $r^2=\rho^2-1$ (which must hold for $\gamma^2>0$) means that all the invariant circles of the pencil are *orthogonal* to the unit (isometric) circle $|z|=1$.

fitted into the above scheme. Here we are to consider the transformation $w=(az+b)/d$, and we obviously require $ad\neq 0$ if an invariant "circle" is to exist (*i.e.* one fixed point must be at infinity). Then an evident *necessary* condition for an invariant proper circle is that $|a/d|=1$, while it is also clearly necessary for the existence of an invariant line that a/d be real; and, in either case,

$$\gamma^2=\frac{(a+d)^2}{4ad}=\frac{1}{4}\left(\frac{a}{d}+2+\frac{d}{a}\right) \quad \text{is real.}$$

Conversely, if γ^2 is real, then

$$a/d=2\gamma^2-1\pm 2\gamma\sqrt{(\gamma^2-1)}$$

has modulus unity if $0\leqslant\gamma^2\leqslant 1$, and is otherwise real.

Supposing first that $a\neq d$ (and therefore $\gamma^2\neq 1$), we can write

$$w-\frac{b}{d-a}=\frac{a}{d}\left(z-\frac{b}{d-a}\right),$$

so that, if $0<\gamma^2<1$, then every circle with centre $b/(d-a)$ is invariant, while otherwise the invariant system is the pencil of lines through this point. If $\gamma=0$ (*i.e.* $a=-d$) then *both* these systems are invariant.

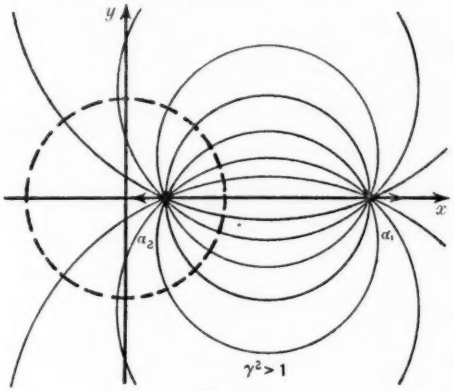


FIG. 5

Finally, if $a=d$, we have $w=z+b/d$, $\gamma^2=1$, and, for $b\neq 0$, the fixed points coincide at infinity; the invariant system is the set of lines parallel to the vector b/d, and may be regarded as a coaxal system with common points at infinity.

It is easily seen that these facts are the natural extensions of our previous results for $c\neq 0$, so, to sum up, if we agree to accept the obvious conventional interpretations of limiting cases, we have the following:

THEOREM. *The bilinear transformation $w=(az+b)/(cz+d)$ has an invariant circle if, and only if, $ad-bc\neq 0$ and $(a+d)^2/(ad-bc)$ is real.*

Further, if these conditions are satisfied, then there is an infinite number of invariant circles, and they form a coaxal system (together with the associated orthogonal system if $a+d=0$; the case $w\equiv z$ is of course exceptional).

M. P. D.

THE STREAM FUNCTION AND THE VELOCITY POTENTIAL FUNCTION

By V. D. Naylor.

Introduction.

The stream function, ψ, is a function specially suited for dealing with two-dimensional flow while the velocity potential, ϕ, is a function which may be used with either two- or three-dimensional flow.

In two-dimensional flow, the chief utility of ψ and ϕ occurs when they satisfy Laplace's equation $\nabla^2\psi=0$ or $\nabla^2\phi=0$, for then, standard solutions of these equations are known which can be fitted to meet the needs of the particular problem under consideration. The connexion between ψ and ϕ is not clearly set out in aerodynamics books ; especially is this the case when it is desired to pass from the study of incompressible flow to the study of compressible flow. In this paper, an attempt is made to make clear the connexion between ψ and ϕ.

Theorem A.

It is proved in the calculus that the necessary and sufficient condition for $P\,dx+Q\,dy$ to be a perfect differential is $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$, P and Q being functions of x and y. When this is so, we may denote $P\,dx+Q\,dy$ by say, dU. Also the equation $P\,dx+Q\,dy=0$ may then be written $dU=0$ and its solution is $U=$ constant. The differential equation is then known as an exact equation.

We next give three theorems (1), (2), (3) concerning ψ and then three theorems (1′), (2′), (3′) concerning ϕ.

(1) *If* $\dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}=0$ *then* ψ *exists and* $u=-\dfrac{\partial\psi}{\partial y}$, $v=\dfrac{\partial\psi}{\partial x}$.

In this case we have $\dfrac{\partial v}{\partial y}=\dfrac{\partial(-u)}{\partial x}$ and, by theorem (A), $v\,dx-u\,dy$ is a perfect differential which we may denote by $d\psi$, so that

$$d\psi=v\,dx-u\,dy.$$

But
$$d\psi=\frac{\partial\psi}{\partial x}\,dx+\frac{\partial\psi}{\partial y}\,dy$$

and hence
$$v=\frac{\partial\psi}{\partial x}\quad\text{and}\quad u=-\frac{\partial\psi}{\partial y}.$$

(2) $\nabla^2\psi=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}$.

To see this, we have

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right)=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.$$

(3) *If* $\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=0$ *then* $\nabla^2\psi=0$.

This is an immediate consequence of (2).

The corresponding sequence of theorems for ϕ is :

(1′) *If* $\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=0$, *then* ϕ *exists and* $u=\dfrac{\partial\phi}{\partial x}$, $v=\dfrac{\partial\phi}{\partial y}$.

In this case we have $\dfrac{\partial u}{\partial y}=\dfrac{\partial v}{\partial x}$ and by theorem (A) $u\,dx+v\,dy$ is a perfect

differential which we may denote by $d\phi$, so that

$$d\phi = u\,dx + v\,dy$$

But

$$d\phi = \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy$$

and hence

$$u = \frac{\partial\phi}{\partial x} \quad \text{and} \quad v = \frac{\partial\phi}{\partial y}.$$

(2′) $\nabla^2\phi = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$.

To see this

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$

(3′) *If* $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$, *then* $\nabla^2\phi = 0$.

This follows directly from (2′).

From the above results it is clear that

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

are related to ψ in the same way that

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

are related to ϕ.

We can combine the foregoing ideas in either of the two ways shown in the following parallel columns.

If $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$,

then ψ exists

and $u = -\dfrac{\partial\psi}{\partial y}, \quad v = \dfrac{\partial\psi}{\partial x}$

and $\nabla^2\psi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$.

If further $\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$,

then $\nabla^2\psi = 0$, then ϕ exists and

and $u = \dfrac{\partial\phi}{\partial x}, \quad v = \dfrac{\partial\phi}{\partial y}$

and $\nabla^2\phi = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$

$= 0$

by virtue of the original assumption.

If $\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0$,

then ϕ exists

and $u = \dfrac{\partial\phi}{\partial x}, \quad v = \dfrac{\partial\phi}{\partial y}$

and $\nabla^2\phi = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y}$.

If further $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$,

then $\nabla^2\phi = 0$, then ψ exists

and $u = -\dfrac{\partial\psi}{\partial y}, \quad v = \dfrac{\partial\psi}{\partial x}$

and $\nabla^2\psi = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$

$= 0$

by virtue of the original assumption.

Remembering the facts that if a fluid is incompressible, then

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{and} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

measures the vorticity ζ, we may express the above mathematical results in the following physical form :

$$\begin{cases} \text{If a fluid is incompressible, then } \psi \text{ exists, } u = -\dfrac{\partial\psi}{\partial y},\ v = \dfrac{\partial\psi}{\partial x} \\ \qquad\qquad \text{and vorticity} = \nabla^2\psi. \\ \text{If, in addition, the motion is irrotational, then } \quad \nabla^2\psi = 0. \end{cases}$$

$$\begin{cases} \text{If the motion is irrotational, then } \phi \text{ exists, } u = \dfrac{\partial\phi}{\partial x},\ v = \dfrac{\partial\phi}{\partial y} \\ \qquad\qquad \text{and } \nabla^2\phi = \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} \\ \text{If, in addition, the fluid is incompressible, then } \nabla^2\phi = 0. \end{cases}$$

Hence for the irrotational motion of an incompressible fluid both ψ and ϕ exist and satisfy Laplace's equation.

But—there are other conditions under which Laplace's equation may be satisfied.

Compressible flow.

It is natural to want to extend the foregoing ideas so as to cover the case of compressible flow.

We have the following three theorems (a), (b), (c) which are the analogues of (1), (2) and (3) :

(a) *If* $\dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) = 0$ *there exists a function* $\bar{\psi}$ *and* $\rho u = -\dfrac{\partial\bar{\psi}}{\partial y},\ \rho v = \dfrac{\partial\bar{\psi}}{\partial x}$

The proof is exactly the same as that given at (1) where u and v are replaced by ρu and ρv and ψ is replaced by $\bar{\psi}$. The bar is introduced to make it clear that ψ and $\bar{\psi}$ do not denote the same physical quantities : ψ denotes the volume per second passing through unit area normal to the flow and $\bar{\psi}$ denotes the mass per second passing through unit area. It is to be noted that the " if " is necessary when (a) is regarded as a purely mathematical theorem : but, is unnecessary from the point of view of physics since $\dfrac{\partial}{\partial x}(\rho u) + \dfrac{\partial}{\partial y}(\rho v) = 0$ is the equation of continuity for steady compressible flow.*

(b) $\nabla^2\bar{\psi} = \dfrac{\partial}{\partial x}(\rho v) - \dfrac{\partial}{\partial y}(\rho u)$.

Proof same as at (2).

(c) *If* $\dfrac{\partial}{\partial x}(\rho v) - \dfrac{\partial}{\partial y}(\rho u) = 0$, *then* $\nabla^2\bar{\psi} = 0$.

This completes the analogy : but in this case no physical interpretation of $\dfrac{\partial}{\partial x}(\rho v) - \dfrac{\partial}{\partial y}(\rho u)$ is apparent in the same way that $\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y}$ measures vorticity.

The corresponding theorems analagous to (1′), (2′), (3′) are given at (a'), (b'), (c') below.

(a') *If* $\dfrac{\partial}{\partial x}(\rho v) - \dfrac{\partial}{\partial y}(\rho u) = 0$, *then a function* $\bar{\phi}$ *exists and* $\rho u = \dfrac{\partial\bar{\phi}}{\partial x},\ \rho v = \dfrac{\partial\bar{\phi}}{\partial y}$.

* Some writers wrongly say that the existence of a stream function implies incompressible flow.

(b′) $\nabla^2\bar{\phi} = \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v)$.

(c′) *If* $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$ (*which of course is always true*), *then* $\nabla^2\bar{\phi} = 0$.

From a mathematical point of view, the analogy between (*a*), (*b*), (*c*), (*a′*), (*b′*), (*c′*) and (1), (2), (3), (1′), (2′), (3′) is complete.

From a physical point of view, the definition of $\bar{\psi}$ implied at (*a*) seems to be the only possible way of introducing a stream function into the study of compressible flow. If we do this, we must continue along the lines indicated by theorem (*b*) : we should stop at theorem (*c*) until we can find a physical interpretation of $\frac{\partial}{\partial x}(\rho v) - \frac{\partial}{\partial y}(\rho u)$. For this last reason it is not advisable to introduce a potential function such as $\bar{\phi}$.

We are thus driven back in compressible flow to the use of $\bar{\psi}$ and ϕ and the work which follows indicate the lines along which $\bar{\psi}$ and ϕ enter into the study of compressible flow.

(α) *Since* $\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$ *a stream function* $\bar{\psi}$ *exists for which*

$$\rho u = -\frac{\partial \bar{\psi}}{\partial y}, \quad \rho v = \frac{\partial \bar{\psi}}{\partial x}.$$

(β) $\nabla^2\bar{\psi} = \frac{\partial}{\partial x}(\rho v) - \frac{\partial}{\partial y}(\rho u)$.

We cannot leave it at this for reasons already given. Expanding, we get

$$\nabla^2\bar{\psi} = \rho\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial \rho}{\partial x} - u\frac{\partial \rho}{\partial y}.$$

If the motion is irrotational, $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ and the equation reduces to

$$\nabla^2\bar{\psi} = v\frac{\partial \rho}{\partial x} - u\frac{\partial \rho}{\partial y}.$$

This reduces to $\nabla^2\bar{\psi} = 0$ *if* $v\frac{\partial \rho}{\partial x} - u\frac{\partial \rho}{\partial y} = 0$.

The condition $v\frac{\partial \rho}{\partial x} - u\frac{\partial \rho}{\partial y} = 0$ would be satisfied if the stream lines are perpendicular to the constant density lines. Hence we may say that

(γ) $\nabla^2\bar{\psi} = 0$ *where the motion is irrotational and the stream lines are perpendicular to the constant density lines* (as would be the case with a simple source).

The way ϕ enters into compressible flow is indicated in the following :

(α′) *If* $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$, *then* ϕ *exists and* $u = \frac{\partial \phi}{\partial x}$, $v = \frac{\partial \phi}{\partial y}$.

As at (1′).

(β′) $\nabla^2\phi = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$.

As at (2′).

For compressible flow,

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0.$$

Thus $$\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=-\left(u\frac{\partial \rho}{\partial x}+v\frac{\partial \rho}{\partial y}\right).$$

and hence $$\nabla^2\phi=-\frac{u}{\rho}\frac{\partial \rho}{\partial x}-\frac{v}{\rho}\frac{\partial \rho}{\partial y}.$$

This is the general equation satisfied by ϕ in compressible flow.

Clearly $\nabla^2\phi=0$ if $\dfrac{v}{u}\cdot\dfrac{\dfrac{\partial \rho}{\partial y}}{\dfrac{\partial \rho}{\partial x}}=-1.$

This will be satisfied if the stream lines coincide with the constant density lines.

Hence

(γ') $\nabla^2\phi=0$ *if the motion is irrotational and if stream lines and constant density lines coincide* (as is the case with a simple vortex). V. D. N.

1765. W. A. Johnston, the Australian bowler and No. 11 batsman, by finishing up with an average of 102 for only once out in 17 innings, gains a lasting place in Wisden's record of freakish achievements.

He lost his wicket on June 6 to Cannings, of Hampshire, who caught and bowled him. That was a kindness, for without that wicket Johnston would not have had an average at all.—*Daily Telegraph*. [Per Mr. H. V. Lowry.]

1766. " Where did you get these figures? " asked the managing director.

" They are Faraday's figures," replied the chief chemist.

" Fetch him," said the director.

" But he has been dead a hundred years," explained the chemist.

Back came the reply. " Don't quote me figures of a man who has been dead a hundred years. We want something up to date."

This story of a factory in 1940 was told in the Commons last night by Mr. Cyril Bence (Lab., Dumbarton, E.).—*News-Chronicle*, 21st July, 1953. [Per Mr. P. Vermes.]

1767. When Prof. Hubert Woyty-Wimmer designed his well-known 20c. definitive for the United Nations some years ago he hit upon the ingenious idea of linking the two hemispheres by a crossed-over ribbon which not only served to join the emblems but at the same time typified the mathematical symbol for *eternity*. Now I note the same motif used in two new issues, one the Portuguese stamp for the 50th anniversary of the Portuguese Automobile Association, the other a special postmark used by Austria to aid Holland.—*Stamp Collecting*, 1st May, 1953. [Per Mr. C. F. Britton.]

1768. At this moment, 5,424 prisoners are sleeping three in a cell. The average cell is 13 ft. by 7 ft. by 9 ft., so that in such cells there is available for each prisoner a little over 4 ft. by 2 ft. by 3 ft.—Letter in *The Times*, 3rd October, 1952.

One-third of 13 ft. by 7 ft. by 9 ft., is not " a little over 4 ft. by 2 ft. by 3 ft." but 13 ft. by 7 ft. by 3 ft.

Even so, this is by no means enough.—Letter in *The Times*, 4th October, 1952. [Per Mr. R. E. Walker.]

1769. Anyone who thinks the fugue a mathematical concoction must revise his opinion in the face of the joyous innovations of this remarkably fine example.—John F. Russell, in analytical notes to Bach's Concerto in C major for Two Pianos and String Orchestra, in the programme of a Hallé Concert, November 1952. [Per Dr. B. H. Neumann.]

MATHEMATICAL NOTES

2384. *An experiment in notation.*

The theory of theta functions contains many expansions of the general form

$$c_0+\frac{c_1x}{1-a_1x}+\frac{c_2x^2}{1-a_1x\,.\,1-a_2x}+\frac{c_3x^3}{1-a_1x\,.\,1-a_2x\,.\,1-a_3x}+\ \dots\ ,$$

where the sequence $a_1, a_2, a_3, \dots$ is given and the coefficients $c_0, c_1, c_2, c_3 \dots$ are usually determined on the basis of the tacit assumption that an expansion of the required form does exist. This note deals with one or two problems in finite expansions of this type; its interest is primarily in the directness of the technique employed.

Given n non-zero constants $a_1, a_2, \dots, a_n$ and a variable y, let y_r denote $y-a_r$. If $P_n(y)$ is a polynomial of degree n, we have by successive division

$$P_n(y)=r_n+y_nP_{n-1}(y),\ P_{n-1}(y)=r_{n-1}+y_{n-1}P_{n-2}(y),\ \dots\ ,$$
$$P_2(y)=r_2+y_2P_1(y),\ P_1(y)=r_1+y_1r_0,$$

where $r_n, r_{n-1}, \dots r_1, r_0$ are constants. Thus we may write

$$P_n(y)=r_n+y_n\{r_{n-1}+y_{n-1}\{r_{n-2}+\dots+y_2\{r_1+y_1\{r_0,$$

omitting the n closing brackets after the constant r_0, which is simply the coefficient of y^n in $P_n(y)$. In particular, there is a unique set of constants $c_0, c_1, \dots, c_n$ such that

$$y^n=c_n+y_n\{c_{n-1}+y_{n-1}\{c_{n-2}+\ \dots\ +y_2\{c_1+y_1\{c_0,$$

and $c_0=1$. Writing $1/x$ for y and p_r for the product

$$(1-a_1x)(1-a_2x)\ \dots\ (1-a_rx),$$

we have established the existence of an identity

$$\frac{1}{p_n}=c_0+\frac{c_1x}{p_1}+\frac{c_2x^2}{p_2}+\dots+\frac{c_nx^n}{p_n}\,.$$

If progress to a limit is under consideration, we must be able to regard $c_0, c_1, \dots c_n$ less as created simultaneously by the set $a_1, a_2, \dots a_n$ than as formed by a_n by modification of the corresponding coefficients related to $a_1, a_2, \dots, a_{n-1}$. Let

$$y^{n-1}=b_{n-1}+y_{n-1}\{b_{n-2}+y_{n-2}\{b_{n-3}+\ -\dots y_2\{b_1+y_1\{b_0.$$

Then

$$y^{n-1}y_n=y_n\{b_{n-1}+y_{n-1}\{b_{n-2}+\dots+y_2\{b_1+y_1\{b_0,$$

and therefore, since $b_0=c_0$

$$a_ny^{n-1}=d_n+y_n\{d_{n-1}+y_{n-1}\{d_{n-2}+\dots+y_2\{d_1,$$

where $d_n=c_n$ and $d_r=c_r-b_r$ if $r<n$. Hence

$$\begin{aligned}a_ny^n&=d_n(a_n+y_n)+y_n\{d_{n-1}(a_{n-1}+y_{n-1})+\dots+y_2\{d_1(a_1+y_1)\\&=a_nd_n+y_n\{d_n+a_{n-1}d_{n-1}+y_{n-1}\{d_{n-1}+a_{n-2}d_{n-2}+\dots+y_2\{d_2+a_1d_1+y_1\{d_1.\end{aligned}$$

On the other hand, since $c_r=b_r+d_r$,

$$y^n=d_n+y_n\{b_{n-1}+d_{n-1}+y_{n-1}\{b_{n-2}+d_{n-2}+\dots+y_2\{b_1+d_1+y_1\{b_0.$$

Hence $d_1=a_n$, and for $n=2, 3, \dots, n$

$$d_r=a_nb_{r-1}-(a_{r-1}-a_n)d_{r-1},$$

whence explicitly

$c_r = b_r + a_n\{b_{r-1} - (a_{r-1} - a_n)\{b_{r-2} - (a_{r-2} - a_n)\{b_{r-3} - \ldots - (a_2 - a_n)\{b_1 - (a_1 - a_n)\{1,$

that is,

$$c_r = b_r + a_n\{b_{r-1} - (a_{r-1} - a_n)b_{r-2} + (a_{r-2} - a_n)(a_{r-1} - a_n)b_{r-3} - \ldots\}.$$

Replacing b_r, c_r by $c_{n-1,r}$ $c_{n,r}$ we have the difference equation for evolving the scheme of coefficients associated with an infinite sequence $a_1, a_2, \ldots$:

$c_{n,r} = c_{n-1,r} + a_n\{c_{n-1,r-1} - (a_{r-1} - a_n)c_{n-1,r-2} + (a_{r-2} - a_n)(a_{r-1} - a_n)c_{n-1,r-3} - \ldots\}$

with $c_{n,0} = 1$ for all values of n, and $c_{n,m} = 0$ if $m > n$. If, for fixed r, $c_{n,r}$ tends to a limit c_r as $n \to \infty$, and if further

$$(c_{n,1} - c_1)\frac{x}{p_1} + (c_{n,2} - c_2)\frac{x^2}{p_2} + \ldots + (c_{n,n} - c_n)\frac{x^n}{p_n} \to 0,$$

then

$$\frac{1}{(1 - a_1x)(1 - a_2x)\ldots} = 1 + \frac{c_1x}{(1 - a_1x)} + \frac{c^2x^2}{(1 - a_1x)(1 - a_2x)} + \ldots .$$

Some form of Tannery's theorem will be essential in any detailed investigation of limits. E. H. N.

2385. *Some forms for the equation of a conic.*

In this note, the following suffix notation is used :

$$S_{11} \equiv ax_1^2 + by_1^2 + cz_1^2 + 2fy_1z_1 + 2gz_1x_1 + 2hx_1y_1,$$
$$S_{12} \equiv ax_1x_2 + by_1y_2 + cz_1z_2 + f(y_1z_2 + y_2z_1) + g(z_1x_2 + z_2x_1) + h(x_1y_2 + x_2y_1),$$
$$S_1 \equiv ax_1x + by_1y + cz_1z + f(y_1z + yz_1) + g(z_1x + zx_1) + h(x_1y + xy_1),$$
$$\xi \equiv ax + hy + gz, \qquad \xi_1 \equiv ax_1 + by_1 + gz_1, \quad \text{etc.}$$

Suppose P_1, P_2 are two points on the conic $S = 0$. Then, for some value of λ,

$$S - \lambda S_1S_2 = 0$$

represents the chord P_1P_2 taken twice, that is, for some constants λ and μ,

$$S - \lambda S_1S_2 \equiv \mu(S_1 + S_2 - S_{12})^2.$$

Polarizing this identity with respect to P_1 and using $S_{11} = 0$,

$$S_1 - \tfrac{1}{2}\lambda S_1S_{12} \equiv \mu(S_1 + S_2 - S_{12})(S_{11} + S_{12} - S_{12}) = 0,$$

so that $\lambda = 2/S_{12}$.

Hence $SS_{12} - 2S_1S_2$ is a perfect square. Considering only the second degree terms in x and y, it follows that

$$S_{12}(ax^2 + 2hxy + by^2) - 2(x\xi_1 + y\eta_1)(x\xi_2 + y\eta_2)$$

is a perfect square, and so

$$(hS_{12} - \xi_1\eta_2 - \xi_2\eta_1)^2 = (aS_{12} - 2\xi_1\xi_2)(bS_{12} - 2\eta_1\eta_2).$$

Thus

$$h^2S_{12}^2 - 2hS_{12}(\xi_1\eta_2 + \xi_2\eta_1) + (\xi_1\eta_2 + \xi_2\eta_1)^2 = abS^2_{12} - 2S_{12}(a\eta_1\eta_2 + b\xi_1\xi_2) + 4\xi_1\xi_2\eta_1\eta_2,$$

that is,

$$CS^2_{12} - 2S_{12}\{\eta_1(a\eta_2 - h\xi_2) + \xi_1(b\xi_2 - h\eta_2)\} = (\xi_1\eta_2 - \xi_2\eta_1)^2$$

or

$$CS^2_{12} - 2S_{12}\{\eta_1(Cy_2 - Fz_2) + \xi_1(Cx_2 - Gz_2)\} = (\xi_1\eta_2 - \xi_2\eta_1)^2 ;$$

thus

$$CS^2_{12} - 2S_{12}\{CS_{12} - C\zeta_1z_2 - F\eta_1z_2 - G\xi_1z_2\} = (\xi_1\eta_2 - \xi_2\eta_1)^2$$

or

$$CS^2_{12} - 2CS^2_{12} + 2S_{12}\Delta z_1z_2 = (\xi_1\eta_2 - \xi_2\eta_1)^2$$

and so

$$CS_{12}^2 - 2\Delta S_{12}z_1z_2 + (\xi_1\eta_2 - \xi_2\eta_1)^2 = 0.$$

Since P_2 lies on $S=0$, the equation of the conic may therefore be written

$$CS_1^2 - 2\Delta S_1z_1z + (\xi_1\eta - \eta_1\xi)^2 = 0.$$

But this equation is of the second degree and its left-hand side is therefore identically λS, for some λ. Polarizing with respect to P_1,

$$-\Delta S_1z_1^2 = \lambda S_1,$$

or

$$\lambda = -\Delta z_1^2,$$

and we have

$$CS_1^2 - 2\Delta S_1z_1z + (\xi_1\eta - \eta_1\xi)^2 + \Delta z_1^2 S \equiv 0. \quad \text{.....(i)}$$

In this identity, equating coefficients of x^2 :

$$C\xi_1^2 + (h\xi_1 - a\eta_1)^2 + az_1^2\Delta = 0.$$

But P_1 is any point on the conic. Hence the equation of the conic may be written

$$C\xi^2 + (h\xi - a\eta)^2 + a\Delta z^2 = 0. \quad \text{.....(ii)}$$

Since the above equation is of the second degree, with coefficient of x^2 equal to Ca^2, we have the identity

$$C\xi^2 + (h\xi - a\eta)^2 + a\Delta z^2 \equiv Ca^2S. \quad \text{.....(iii)}$$

From this identity we easily deduce the well-known result

$$b\xi^2 - 2h\xi\eta + a\eta^2 + \Delta z^2 \equiv CS. \quad \text{.....(iv)}$$

Again, from (i), equating coefficients of z^2,

$$C\zeta_1^2 - 2\Delta\zeta_1z_1 + (f\xi_1 - g\eta_1)^2 + \Delta cz_1^2 = 0,$$

so that the equation of the conic can be written

$$C\zeta^2 - 2\Delta\zeta z + (f\xi - g\eta)^2 + \Delta cz^2 = 0. \quad \text{.....(v)}$$

Or again, from (i), equating coefficients of xz,

$$2C\xi_1\zeta_1 - 2\Delta\xi_1z_1 + 2(h\xi_1 - a\eta_1)(f\xi_1 - g\eta_1) + 2\Delta gz_1^2 = 0,$$

so that the equation of the conic may be written

$$C\xi\zeta - \Delta\xi z + (h\xi - a\eta)(f\xi - g\eta) + \Delta gz^2 = 0. \quad \text{.....(vi)}$$

Other similar identities may be obtained by the above method, starting with identities corresponding to (i) above, considering the terms of the second degree in y and z, or z and x in the expression $SS_{12} - 2S_1S_2$.

F. M. GOLDNER.

2386. *Hypotenuse numbers.*

Mr. W. D. Evans' Note 2208 suggests a further interesting fact concerning hypotenuse numbers. When two hypotenuse numbers are multiplied together, their product is the hypotenuse of two distinct prime right-angled triangles ; this is proved incidentally in the course of the proof of the theorem Mr. Evans enunciates. Conversely, if an hypotenuse number can be expressed as the hypotenuse of two distinct right-angled triangles, it must be the product of two hypotenuse numbers.

To prove this we start with the well-known fact that this hypotenuse number can be written in the form $a^2 + b^2$, where a is odd and b even. Since it is the hypotenuse of two prime triangles, it is also expressible as $c^2 + d^2$, where c is odd and d even. We may suppose $a > c$.

Now $$a^2+b^2=c^2+d^2$$

and so $$(a+c)(a-c)=(d+b)(d-b). \quad \text{.....(i)}$$

Since a and c are both odd, $a+c$, $a-c$ are both even, and similarly $b+d$, $d-b$ are both even. Let x be the highest common factor of $\frac{1}{2}(a+c)$, $\frac{1}{2}(d-b)$. Then

$$\tfrac{1}{2}(a+c)=xm, \quad \tfrac{1}{2}(d-b)=xn, \quad m, n \text{ co-prime},$$

and if y is the highest common factor of $\frac{1}{2}(a-c)$, $\frac{1}{2}(d+b)$,

$$\tfrac{1}{2}(a-c)=yn', \quad \tfrac{1}{2}(d+b)=ym', \quad m', n' \text{ co-prime}.$$

Then, from (i), we have

$$xm \,.\, yn' = xn \,.\, ym'$$

or $$m/n = m'/n',$$

and since m, n are co-prime, and m', n' are co-prime, we have

$$m=m', \quad n=n'.$$

Thus $$\tfrac{1}{2}(a+c)=xm, \quad \tfrac{1}{2}(a-c)=yn,$$

so that $$a=xm+yn, \quad c=xm-yn\,;$$

and also $$\tfrac{1}{2}(d+b)=ym, \quad \tfrac{1}{2}(d-b)=xn,$$

so that $$d=ym+xn, \quad b=ym-xn.$$

Thus $$a^2+b^2\,(=c^2+d^2)=x^2m^2+y^2n^2+2xymn+y^2m^2+x^2n^2-2ymxn$$
$$=(x^2+y^2)(m^2+n^2).$$

Now, since a^2+b^2 is odd, both x^2+y^2 and m^2+n^2 must be odd, implying that one element in each pair is odd, and one even. Moreover, it is impossible that either pair should have a common factor, since a, b are co-prime. Hence x^2+y^2 and m^2+n^2 are both hypotenuse numbers. N. H. SMITH.

2387. *A note on the reversion of numbers.*

1. Parameswaran has shown in the *Mathematical Gazette* of May 1947 that if a and b are two groups, each of k digits, then

$$ab-ba=(a-b)\,.\,(9)_k \quad \text{.....(1)}$$

where $(9)_k = 999\ldots$ to k digits.

Examples.

(*a*) $$53-35=2\,.\,9$$
$$68-86=(-2)\,.\,9$$

(*b*) $$\overline{23}\;\overline{22}-\overline{22}\;\overline{23}=99$$
$$\overline{12}\;\overline{13}-\overline{13}\;\overline{12}=(-1)\,.\,99$$

(*c*) $$\overline{342}\;\overline{152}-\overline{152}\;\overline{342}=190\,.\,999$$
$$\overline{003}\;\overline{975}-\overline{975}\;\overline{003}=(-972)\,.\,999.$$

This can be shown easily since if (a_j, a_i) represent a two digital number, then the difference of the number and its reversion is

$$(a_j, a_i)-(a_i, a_j)=10a_j+a_i-10a_i-a_j$$
$$=9(a_j-a_i) \quad \text{.....(2)}$$

[where j and i may assume independently the values, 1, 2, ... , 9, 0].

2. Consider the case of the number of three digits represented by

$$(a_k, a_j, a_i). \quad [k=1, 2, \ldots, 9, 0]$$

Then

$$(a_k, a_j, a_i) - (a_i, a_j, a_k) = 100a_k + 10a_j + a_i - 100a_i - 10a_j - a_k$$
$$= 99(a_k - a_i), \quad \ldots\ldots(3)$$

i.e. the difference between a three-digital number and its reversion is equal to 99 times the difference of its extreme integers.

3. *Number of four digits.*

Let (a_l, a_k, a_j, a_i) be a number of four digits, then the difference between this number and its reversion becomes

$$(a_l, a_k, a_j, a_i) - (a_i, a_j, a_k, a_l) = 1000a_l + 100a_k + 10a_j + a_i - 1000a_i - 100a_j - 10a_k - a_l$$
$$= 999(a_l - a_i) + 90(a_k - a_j). \quad \ldots\ldots(4)$$

Thus the difference between a number of four digits and its reversion is 999 times the difference of the extremes of the number plus 90 times the difference of the second and third integers.

4. *Case of five digits.*

$n = 5$, and number is $(a_m, a_l, a_k, a_j, a_i)$

$$(a_m, a_l, a_k, a_j, a_i) - (a_i, a_j, a_k, a_l, a_m) = 9999(a_m - a_i) + 990(a_l - a_j) \quad \ldots\ldots(5)$$

In general, it follows that if there be n digits in a given number, the expression for the difference between the number and its reversion will contain

$$[(2n-1) + (-1)^n]/4 \text{ terms} \quad \ldots\ldots(6)$$

(n being odd or even).

More concisely, if a_n denote the value of the digit in the nth place and a_{n-1} denotes the value of the digit in the $(n-1)$th place, then the difference is given by the series

$$(10^{n-1} - 1)(a_n - a_1) + (10^{n-2} - 10)(a_{n-1} - a_2) + (10^{n-3} - 10^2)(a_{n-2} - a_3) \quad \ldots(7)$$

(where, for any n, all coefficients are + ve).

Substitution for $n = 2$, 3, 4 and 5 will give the equations of (2), (3), (4) and (5).

N. J. C. PERES.

2388. *The continued fraction for* $F(a, 1\ ;\ c\ ;\ t)$.

It is well known that the hypergeometric series

$$F(a, 1\ ;\ c\ ;\ t) = \sum_{s=0}^{\infty} \frac{a(a+1)\ldots(a+s-1)}{c(c+1)\ldots(c+s-1)} t^s \quad \ldots\ldots(1)$$

can be represented as a continued fraction (due to Gauss), and

$$F(a, 1\ ;\ c\ ;\ t) = \frac{1}{1-}\ \frac{b_1 t}{1-}\ \frac{b_2 t}{1-}\ \frac{b_3 t}{1-}\ \frac{b_4 t}{1-}, \quad \ldots\ldots(2)$$

where

$$b_{2s+1} = \frac{(a+s)(s+c-1)}{(c+2s-1)(c+2s)}, \quad b_{2s+2} = \frac{(1+s)(c-a+s)}{(c+2s)(c+2s+1)}, \quad s = 0, 1, 2, \ldots .$$

The conditions of convergence and equivalence of (1) and (2) may be found, for example, in Wall*, pp. 335—342. We shall use N_s and D_s for the numerator and denominator respectively of the sth convergent of (2).

Consider $\quad G_s(t) = D_s(t)F(a, 1\ ;\ c\ ;\ t) - N_s(t)$

or $\quad G_s = D_s F(a, 1\ ;\ c\ ;\ t) - N_s. \quad \ldots\ldots(3)$

* *Analytic Theory of Continued Fractions*, H. S. Wall ; (Van Nostrand, 1948).

Then $F(a, 1\ ;\ c\ ;\ t)$ and D_s satisfy a hypergeometric-type differential equation, for

$$D_{2s} = F(-s, 1-a-s\ ;\ 2-c-2s\ ;\ t),$$
$$D_{2s+1} = F(-s, -a-s\ ;\ 1-c-2s\ ;\ t), \quad \ldots\ldots(4)$$
$$s = 0, 1, \ldots .$$

Now

$$F(a, 1\ ;\ c\ ;\ t) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{x^{a-1}(1-x)^{c-a-1}}{1-xt}\, dx,$$

provided $|t| < 1$, $c > a > 0$ (apart from particular cases). Using this, a theorem due to Sherman (*Trans. Am. Math. Soc.*, Vol. 35, 1933, pp. 64—87) supplies the differential equation satisfied by N_s. It will now be found that G_s satisfies a second order differential equation, from which it can be shown that

$$G_{2s} = b_1 b_2 \ldots b_{2s} t^{2s} F(s+a, s+1\ ;\ 2s+c\ ;\ t), \quad \ldots\ldots(5)$$
$$G_{2s+1} = b_1 b_2 \ldots b_{2s+1} t^{2s+1} F(s+a+1, s+1\ ;\ 2s+c+1\ ;\ t). \quad \ldots\ldots(6)$$

Hence

$$F(a, 1\ ;\ c\ ;\ t) - \frac{N_{2s}}{D_{2s}} = b_1 b_2 \ldots b_{2s} t^{2s} \frac{F(s+a, s+1\ ;\ 2s+c\ ;\ t)}{F(1-s-a, -s\ ;\ 2-2s-c\ ;\ t)}, \quad \ldots\ldots(7)$$

$$F(a, 1\ ;\ c\ ;\ t) - \frac{N_{2s+1}}{D_{2s+1}} = b_1 b_2 \ldots b_{2s+1} t^{2s+1} \frac{F(s+a+1, s+1\ ;\ 2s+c+1\ ;\ t)}{F(-s-a, -s\ ;\ 1-2s-c\ ;\ t)}. \quad \ldots(8)$$

with $|t| < 1$, $c > a > 0$, ignoring the cases $|t| = 1$.

(7) and (8) do not seem to be widely quoted in the literature. We may deduce from these a property concerning the products of hypergeometric functions. For it is known (see Wall, pp. 338–339) that

$$G_{2s} = \frac{b_1 b_2 \ldots b_{2s} t^{2s}}{D_{2s+1} - t b_{2s+1} D_{2s} P_{2s+1}}, \quad \ldots\ldots(9)$$

$$G_{2s+1} = \frac{b_1 b_2 \ldots b_{2s+1} t^{2s+1}}{D_{2s+2} - t b_{2s+2} D_{2s+1} P_{2s+2}}, \quad \ldots\ldots(10)$$

where

$$P_{2s+1} = \frac{F(a+s+1, s+1\ ;\ c+2s+1\ ;\ t)}{F(a+s, s+1\ ;\ c+2s\ ;\ t)},$$

$$P_{2s+2} = \frac{F(a+s+1, s+2\ ;\ c+2s+2\ ;\ t)}{F(a+s+1, s+1\ ;\ c+2s+1\ ;\ t)}.$$

Hence, from (5) – (6) and (9) – (10), we have the relations

$$F(-a-s, -s\ ;\ 1-c-2s\ ;\ t) F(a+s, s+1\ ;\ c+2s\ ;\ t)$$
$$- t b_{2s+1} F(1-a-s, -s\ ;\ 2-c-2s\ ;\ t) F(a+s+1, s+1\ ;\ c+2s+1\ ;\ t) = 1,$$
$$F(-a-s, -s-1\ ;\ -c-2s\ ;\ t) F(a+s+1, s+1\ ;\ c+2s+1\ ;\ t)$$
$$- t b_{2s+2} F(-a-s, -s\ ;\ 1-c-2s\ ;\ t) F(a+s+1, s+2\ ;\ c+2s+2\ ;\ t) = 1.$$

L. R. SHENTON.

2389. *Inverse circular functions.*

The following point in the treatment of the inverse circular functions seems to have been overlooked by most writers of textbooks on trigonometry, calculus, and the theory of functions. Having defined the inverse sine, cosine, and tangent as many-valued functions, and having then specified the principal branches of these functions by the inequalities

$$-\frac{\pi}{2} \leqslant \arcsin x \leqslant \frac{\pi}{2}, \quad 0 \leqslant \arccos x \leqslant \pi, \quad -\frac{\pi}{2} < \arctan x < \frac{\pi}{2},$$

many writers content themselves with the remark that the three remaining inverse circular functions may be dealt with in the same way.

Whereas obviously the range of arc sec should be chosen the same as that for arc cos, and the range of arc cosec the same as that for arc sin, it is not obvious whether the principal branch of arc cot x should be defined by

$$-\frac{\pi}{2} < \text{arc cot } x \leqslant \frac{\pi}{2} \quad \text{or by} \quad 0 < \text{arc cot } x < \pi.$$

The first definition has the advantage that of the infinity of values of arc cot x, that one is chosen for arc cot x which is the numerically least. (Arc cot 0 has two numerically equal least values of which the positive one is chosen as the value of arc cot 0.) The definition is in this respect in line with that of principal value for the remaining five inverse circular functions and for the amplitude θ of a complex number, $-\pi < \theta \leqslant \pi$.

On the other hand, the second definition makes arc cot x a continuous function for all x. Of the formulae,

$$\text{arc cot } x = \text{arc tan } \frac{1}{x}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\text{arc cot } x + \text{arc tan } x = \frac{\pi}{2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

either the one or the other breaks down for part of the x-range.

With the first definition, (1) holds for all $x \neq 0$, (2) holds for $x \geqslant 0$ and has to be supplemented for $x < 0$ by

$$\text{arc cot } x + \text{arc tan } x = -\frac{\pi}{2}.$$

With the second definition, (2) holds for all x, whereas (1) now holds for $x > 0$ only, and has to be supplemented for $x < 0$ by

$$\text{arc cot } x = \pi + \text{arc tan } \frac{1}{x}.$$

The second definition seems more suitable and is, in fact, followed by most writers who care to define arc cot at all. Whatever choice be made, it is desirable that an explicit definition of arc cot should be given. Moreover (and this applies to the remaining inverse circular functions as well) these definitions should be applied consistently. In this respect, most textbooks fail (with a few notable exceptions, like Durell and Robson's *Advanced Trigonometry*). The following examples are all taken from well-known high school and university textbooks :

(*a*) $\text{arc cos } \dfrac{1-x^2}{1+x^2} = \text{arc tan } \dfrac{2x}{1-x^2} \quad (\equiv \alpha, \text{ say}).$

R.H.S. should read : $\pi - \alpha$ when $x < -1$, $-\alpha$ when $-1 < x \leqslant 0$, α when $0 \leqslant x < 1$, and $\pi + \alpha$ when $x > 1$.

(*b*) $\text{arc sin } a + \text{arc sin } b = \text{arc sin } (a\sqrt{1-b^2} + b\sqrt{1-a^2}) \quad (\equiv \beta, \text{ say}).$

R.H.S. should read : $-\pi - \beta$ when $a, b < 0$ and $a^2 + b^2 \geqslant 1$, β when either $ab \leqslant 0$ or $ab > 0$ and $a^2 + b^2 \leqslant 1$, $\pi - \beta$ when $a, b > 0$ and $a^2 + b^2 \geqslant 1$.

(*c*) $\text{arc cos } a + \text{arc cos } b = \text{arc cos } (ab - \sqrt{1-a^2} \,.\, \sqrt{1-b^2}) \quad (\equiv \gamma, \text{ say}).$

R.H.S. should read : $2\pi - \gamma$ when $a + b \leqslant 0$, γ when $a + b \geqslant 0$.

(*d*) $\text{arc sin } (\sqrt{2} \sin \theta) + \text{arc sin } \sqrt{\cos 2\theta} = \tfrac{1}{2}\pi.$

R.H.S. should read : $\tfrac{1}{2}\pi$ or arc sin $(2 \cos 2\theta - 1)$, depending on the value of θ.

(*e*) $\tan\left\{\frac{1}{2}\arcsin\frac{2x}{1+x^2}+\frac{1}{2}\arccos\frac{1-y^2}{1+y^2}\right\}=\frac{x+y}{1-xy}$.

(*f*) To find the solution(s) of the equation

$$\arctan 2x+\arctan 4x=\arctan 3 \quad \text{(Answ. } \tfrac{1}{4} \text{ or } -\tfrac{1}{2}\text{)}.$$

To find conditions on θ corresponding to the alternative answers in (*d*), and to supply the correct answers to (*e*) and (*f*) is left as an exercise to the reader.

H. W.

2390. *Some geometrical properties of the centre of pressure.*

Introduction.

The determination of the centre of pressure of a plane lamina immersed in heavy homogeneous liquid is found in text-books on Hydrostatics. Further, some text-books discuss the " body locus " of the centre of pressure (*i.e.* the locus of the centre of pressure relative to the lamina) in the cases (*a*) when the lamina has its plane vertical and rotates in its own plane about its centre of gravity G which is fixed, the lamina remaining totally immersed throughout, (*b*) when the lamina has its plane vertical, and descends vertically without rotation.

Very little appears to be said in the text-books, however, about the corresponding " space locus " of the centre of pressure (*i.e.* the locus of the centre of pressure in space) in the cases (*a*), (*b*) above.

In this article, the space locus of the centre of pressure is discussed and its relationship with the body locus of the centre of pressure for cases (*a*), (*b*).

In the figure below, KL represents the line of intersection of the effective free surface with the plane of the lamina. G is the centre of gravity of the lamina at depth h below KL. Gx, Gy are the principal axes of inertia of the lamina at G (in the plane of the lamina); GX, GY are axes at right angles in the plane of the lamina, GX being vertically downwards and intersecting KL in M. Thus, if G is fixed, Gx, Gy are axes fixed in the lamina, while GX, GY are axes fixed in space.

Suppose the centre of pressure P has position (x, y) referred to Gx, Gy as axes and position (X, Y) referred to GX, GY as axes, so that when $\angle xGX=\theta$,

$$\left.\begin{aligned} X&=x\cos\theta-y\sin\theta\\ Y&=x\sin\theta+y\cos\theta\end{aligned}\right\}. \qquad \text{.....(i)}$$

Then we may easily prove that

$$x=a\cos\theta, \quad y=-b\sin\theta, \qquad \text{.....(ii)}$$

where $a=k_y^2/h$; $b=k_x^2/h$; k_x, k_y being the radii of gyration of the lamina about Gx, Gy respectively.

Hence (i), (ii) give

$$X=\tfrac{1}{2}(a+b)+\tfrac{1}{2}(a-b)\cos 2\theta,$$
$$Y=\tfrac{1}{2}(a-b)\sin 2\theta.$$

Case (a). G is fixed, so that h is constant, *i.e.* a and b are constant. Therefore body locus of P is the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and space locus of P is the circle

$$\left(X-\frac{a+b}{2}\right)^2+Y^2=\left(\frac{a-b}{2}\right)^2$$

which has its centre O at a depth $\frac{a+b}{2}$ below G, and which touches the auxiliary

circle of the body locus, namely $X^2+Y^2=a^2$, at B, and the "minor axis" circle $X^2+Y^2=b^2$ at A.

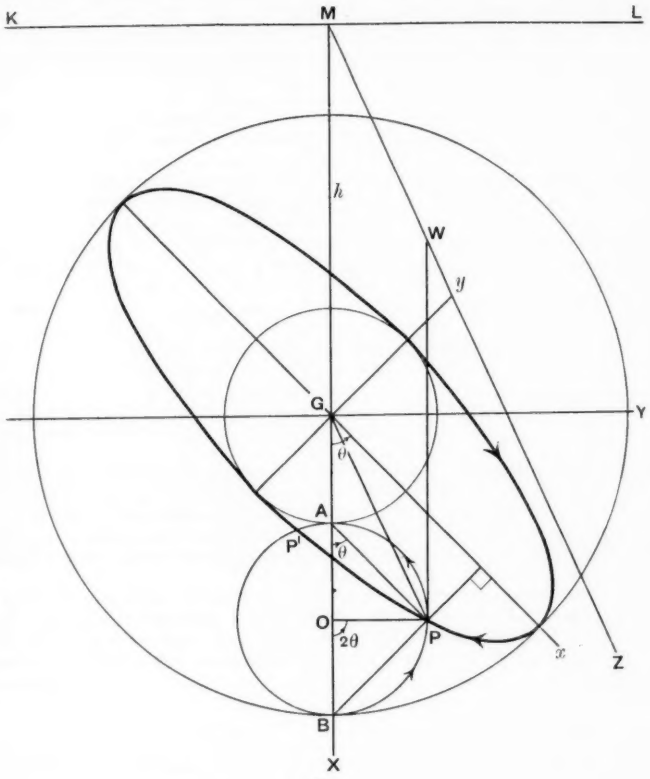


FIG.

Space locus of P is circle centre O.
Body locus of P is ellipse centre G.

In the figure, which is drawn for the case $a>b$, we observe that as the lamina rotates in an anti-clockwise direction, P moves along its body locus in a clockwise direction and along its space locus in an anti-clockwise direction. The arrows in the figure indicate the directions of travel of P.

(For, if $a>b$, and θ is acute, then y is less than zero and x decreases as θ increases; also, Y is greater than zero and X decreases as θ increases.)

We notice that for one complete revolution of the lamina, the body locus is described *once*, while the space locus is described *twice*. The interpretation of this, of course, is that for a given position of P in *space*, the lamina has two possible positions, corresponding to the angles θ, $\theta+\pi$.

It will be noted that the space locus and body locus of P intersect in two real points P and P'. For $a > b$, and θ acute, $Y > 0$; therefore P is the centre of pressure. P' appears to have no special significance, except that it would be the centre of pressure in the case $a < b$.

From the figure we may deduce various geometrical results. Join PA, PO, PB. Then (i) $\angle POB = 2\theta$, (ii) BP produced meets Gx at right angles, (iii) AP is parallel to Gx, so that we may obtain various elementary constructions to determine P.

(For instance, construct the points A, B at distances b, a respectively vertically below G. Through A draw a line parallel to Gx. Then P is the foot of the perpendicular from B on to this line.)

In the special case $b = a$, the space locus of P is the " point " circle

$$(X - a)^2 + Y^2 = 0,$$

i.e. P is fixed in space at a depth a vertically below G.

Case (b). θ fixed, G moving vertically.

Join PG.

Now, $AP = AB \cos\theta = (a - b)\cos\theta$, $GA = b$, therefore

$$\frac{(a-b)\cos\theta}{\sin PGA} = \frac{b}{\sin(\theta - PGA)},$$

therefore

$$\frac{\sin PGA}{\sin(\theta - PGA)} = \frac{(a-b)}{b}\cos\theta = \frac{k_y^2 - k_x^2}{k_y^2}\cos\theta,$$

which is independent of h. Hence $\angle PGA$ is constant. Further, if h is increased, PG decreases inversely proportionally to h. Thus the body locus is a straight line. This result may be found in text-books in the following form, the usual proof being a statical one. " The effect of lowering a plane area to a greater depth, without rotation, is to displace the centre of pressure along a straight line in the area towards the centre of gravity."

To discuss the space locus of P, we may proceed as follows. Through M draw MZ parallel to GP and PW parallel to GX meeting MZ in W. Then, since θ is fixed, MZ is fixed in direction. Now $PW = GM = h$, and $PG \propto 1/h$.

Therefore $PG \,.\, PW$ is constant, *i.e.* space locus of P is a hyperbola whose asymptotes are MX and MZ.

H. W. PARSONS.

2391. *Johann Kepler : Paralipomena ad Vitellionem.*

By the kindness of Professor E. H. Neville I have been able to examine this book at leisure, and it occurred to me that a brief account of it might be of interest.

Kepler, born at Weil in Wurtemburg, 27th December, 1571, died 15th November, 1630, became assistant to Tycho Brahe, who was " Imperial Mathematician " to the Emperor Rudolph II at Prague. Shortly before Brahe died in 1601, he asked Kepler to complete some tables on planetary motions which had occupied him for some years ; thus Kepler became possessed of a remarkably fine collection of astronomical observations, and in due time succeeded to Brahe's post.

Vitellio, a Polish natural philosopher, was one of the few mediaeval students of refraction, on which he published some not very important work about 1260. This was republished later in the forms :

(i) *περὶ ὀπτικῆς.* De natura, ratione, et projectione radiorum visus, luminum, colorum atque formarum quam vulgo Perspectivam vocant, libri X (Norimbergae, 1535).

(ii) *Thuringopolonus.* Opticae libri decem, instaurati, figuris novis illustrati—infinitisque erroribus—expurgati a Federico Risnero (Basileae, 1572).

Clemens Bäumker's " Philosophie des Mittelalters ", vol. III (Münster, 1908) devotes some 650 pages to " Witelo, ein Philosoph und Naturforscher des XIII Jahrhunderts ".

In 1604, Kepler published *Paralipomena ad Vitellionem*, supplementing and correcting the work of Vitellio, Alhazen and others ; the dedication is to his patron " the ever august Roman Emperor, Rudolph II ". Actually, the five chapters of the *Paralipomena* occupy 221 pages, the remaining 228 pages, numbered sequentially, being headed *Astronomiae pars Optica*. Subjects treated are : the nature and properties of light ; colour ; reflection and refraction of light of stars ; diminution of moon's diameter in solar eclipses—this is a continuation of Brahe's work ; measurement of refractions ; errors due to faulty instruments and incorrect sight ; dimensions of sun and moon ; light, position and motion of stars and comets ; age of moon ; parallaxes ; eclipses ; occultation of stars ; twilight ; phases of moon ; altitude of sun and stars ; anatomy of human eye and binocular vision. Occasionally philosophical discussions intrude : thus the sphere, having a centre, interior and surface, is an emblem of the Trinity, and there is a long discussion of the reasons why the human eye is placed at the side of the nose and not at the top of the head (" because man is a social and political animal "). Kepler details an experiment undertaken to explain why the sun's image in a camera obscura is round, although the light is admitted through a rectangular slit. Among numerous results obtained are : the speed of light is infinite ; the diameter of the sun in perigee is 31′, in apogee 30′ ; in July 1600 the diameter of the moon was between 31′ 12″ and 29′ 30″ ; the height of the atmosphere is $5{\cdot}578 \times 10^{-4}$ of the earth's radius, the latter being 860 " German miles " ; while binocular vision is perfect, a correction should be applied in the case of the astronomer who uses only one eye. There are some well printed parallactic and refraction tables (with a long list of corrigenda). Values of circular functions are printed thus : 50° secans 155572 ; 33° sinus 54464 ; the decimal for $\sqrt{3}$ is given incorrectly, though this may be a misprint.

The pure mathematician will chiefly be interested in a short note of some four pages on conics (introduced in the chapter on measurement of refractions, in which Kepler makes use of the hyperbola), and in his parallel postulate, which, closely translated, is : " let it be granted that two light rays emanating from a luminous point, even though in reality they meet in this their source, may yet be treated as lines sensibly equidistant if their length bears a very great ratio to that of a line connecting them ". The particular application of this is that a point source at a very great distance from a sphere will sensibly illuminate a hemisphere ; see also below on the parabola.

The trend of Kepler's thought may be indicated by the following summary of his note on conics.

Cones may be right-angled, acute-angled or obtuse-angled, right or oblique. Their sections fall into five classes, the straight line, circle, parabola, hyperbola, ellipse. " Speaking rather analogically than geometrically " the transition is from the straight line through the hyperbola to the parabola, then through the ellipse to the circle. The straight line is the most obtuse of all hyperbolas, the parabola the most acute ; the parabola is the most acute of all ellipses, the circle the most obtuse. Thus the parabola has two infinite limiting affinities, the hyperbola and straight line, and two finite limiting affinities, the ellipse and circle : it is itself a mean between these, " for though it is infinite it seeks to attain finiteness since the further it is produced the more parallel to itself it becomes, and does not expand its arms (so to speak) like the hyperbola, but draws them together in an infinite embrace ". Thus there are

two opposite bounds, the circle and straight line, the one perfect curvature, the other perfect straightness; between these are included the hyperbola, parabola and ellipse, which participate in both curvature and straightness, the parabola equally, the hyperbola more in respect to straightness, the ellipse more in respect of curvature: wherefore the more the hyperbola is produced, the closer it approximates to the straight line, its asymptote, but the further the ellipse is produced the more it strives after circularity and finally re-enters into itself. The parabola holds a middle position and is always more curved than the hyperbola, but always straighter than the ellipse. As all straight lines are similar, and all circles similar, so all parabolas are similar, differing only in magnitude.

Connected with these curves there are certain points which have a recognized definition but no name unless we make use of the definition or of some property. Lines drawn from one of these points to the points of contact of tangents to the curve make angles equal to those made by joining the other of these points to the same points of contact. " On account of [their connection with] light, and with our eyes directed towards mechanics we shall call them Foci: we should have called them ' centres ', since they lie in the axes of the sections, were it not that authors dealing with the ellipse and hyperbola have appropriated that name for another point." There is one focus in the circle, which coincides with the centre: in the ellipse there are two foci equidistant from the centre. In the parabola one focus lies within the curve; the other, whether inside or outside the curve, is to be conceived as lying on the axis at an infinite distance from the first, so that lines drawn from this " blind " focus to meet the curve in any point are parallel to the axis. In the hyperbola the more obtuse the curve the nearer is the external focus to the internal one, and the one which is external to one branch is internal to the other.

By analogy, it follows that in the straight line both foci coincide on the line —we make this statement not in order to use the foci but to complete the analogy, and we postulate one focus as in the circle. Thus in the circle the focus is at the centre as far distant as possible from the nearest part of the circumference; in the ellipse it recedes rather less and in the parabola much less, while in the straight line it recedes as little as possible from it. Thus in the bounding cases of the circle and straight line the foci coincide, in the former at the greatest distance from the curve, in the latter the focus lies on the line. In the mean curve, the parabola, the two foci are separated by an infinite distance; in the flanking ellipse and hyperbola two real foci are separated by a finite distance—in the ellipse both lying within the curve, in the hyperbola one within and one without.

The line through the focus perpendicular to the axis is called the " chorda " and that part of the axis intercepted between the focus and the nearest part of the curve " sagitta ". In the circle the sagitta is equal to the semichorda: in the ellipse the semichorda $>$ the sagitta $>$ one-fourth of the chorda. In the parabola, as Vitellio has shown, the sagitta is equal to one-fourth of the chorda: in the hyperbola the sagitta $<$ one-fourth of the chorda, and as the curve becomes more obtuse the sagitta diminishes until it vanishes in the straight line, the chorda becoming infinite. " For indeed I find pleasure in analogies, my most reliable instructors, repositories of all the secrets of nature, especially to be looked for in geometry, where they comprehend an infinity of special cases, however incongruously formulated, and place the whole essence of any matter clearly before our eyes."

The note ends with a description of the usual " pins and thread " method of describing the ellipse and two similar but not very practical ones of drawing the hyperbola and parabola. B. A. S.

REVIEWS

A Mathematician's Miscellany. By J. E. Littlewood. Pp. vii, 136. 15s. 1953. (Methuen)

For many readers this is the book of the year. Did the author intend that it should be published before the coronation to provide problems to beguile the hours of waiting in Westminster Abbey and the streets of London? Did it then occur to him that one of the officers of state might become engrossed in, say, the lion and man problem (p. 135) and fail to play his part in the ceremony; and did the possibility lead him to hold up the proof-sheets? Or perhaps he pictured all the duchesses with their coronets awry, and each laughing at the others (cf. p. 3); is the reader persuaded that one of them would deduce that her own coronet must be awry, or will he ask himself whether Professor Littlewood may be pulling his leg?

Readers of the 300th number of the *Gazette* will remember the article on Large Numbers (here reprinted). The author has been described as "a legendary rather than a public figure" (*Nature*, Vol. 165, p. 341) and it is likely that he is known by sight to few members of the Mathematical Association except his former pupils. Some members may have seen without identifying him as a figure on a perpendicular rock-face (? perpendicular (1930) = overhanging (1900) = slope of 70°, p. 44).

His work was described in the following terms by the President of the Royal Society in 1943 when he received the Sylvester medal (*Proc. R. S.* 182 (A), p. 231):

"Among mathematicians the name of Littlewood will always be associated with that of G. H. Hardy for the wonderful example of perfect teamwork in their fashioning of a new and rare instrument of mathematical precision during the second and third decades of the twentieth century. Into the theory of numbers, that subject where problems are so easy to state but so hard to prove, an entirely new spirit was infused when the methods of analysis were brought to bear upon it. The credit for attaining the highest results, for perfecting the methods, and for inspiring an enthusiastic following among so many of the ablest mathematicians at home and abroad, all belongs to Hardy and to Littlewood. Littlewood, who is by age the junior in this remarkable partnership, brought his own magnificent contribution as Hardy himself has wholeheartedly testified. Littlewood, on Hardy's own estimate, is the finest mathematician he has ever known. He was the man most likely to storm and smash a really deep and formidable problem: there was no one else who could command such a combination of insight, technique and power."

I quote this as sufficient evidence that, when Professor Littlewood makes a collection from his note-books to entertain the amateur, it will be high-class entertainment. There are several sections which are for the professional only, one of the most tantalizing being "The Zoo"; the crocodile's teeth and the hedgehog's spines are picturesque illustrations of pieces of high-brow mathematical analysis. But most of the book is within the range of the amateur in mathematics, and a number of pages (e.g. Cross-purposes, Misprints, etc.) are independent of mathematics.

There are eighteen pages of autobiography (A Mathematical Education) ending with the sentence "I soon began my 35-year collaboration with Hardy". Of himself since 1911 the author gives only fleeting glimpses (talking, tongue in cheek, as a subaltern, to a colonel—ideas visiting him on walks, etc.).

A statement by H. Bohr of the axioms of the Hardy-Littlewood partnership has recently appeared in print and, as it may be inaccessible to many readers, I reproduce it.

(1) When one wrote to the other, it was completely indifferent whether what they wrote was right or wrong.

(2) When one received a letter from the other he was under no obligation to read it, let alone to answer it.

(3) Although it did not really matter if they both thought about the same detail simultaneously, still it was preferable that they should not do so.

(4) It was quite indifferent if one of them had not contributed the least bit to the contents of a paper under their common name.

J. C. B.

Mathematik als Wissenschaft, Kunst und Macht. By H. HASSE. Pp. 34. DM. 4.80. 1952. (Verlag für angewandte Wissenschaften, Wiesbaden)

Here is Professor Hasse's inaugural lecture delivered at Hamburg University in 1951. Such a lecture presents a great problem to a mathematician, for what can he say that will be of interest and reasonably intelligible to the audience he has in mind or can expect? Many of his hearers, even if mathematicians, may not be familiar with his special field, and would soon be lost if he ventured even a little way into it. Moreover, he may not hold the attention of the non-mathematician if this theme is too commonplace.

Hasse has found an excellent solution to this problem. He has produced a lecture intelligible and of considerable interest not only to anyone with the merest smattering of mathematical knowledge, but also to the skilled mathematician. The lecture is really a declaration of faith. It might be more aptly called the confession of a mathematician, and that from one who has played a foremost part in the development of algebra and number-theory, not only by his original contributions, but also by his numerous expository articles and books. It is not in any sense of the word an apology. He considers there is no need to apologize for being a mathematician even of the purest kind. On the contrary there is every reason to be proud of a science with a history of several thousand years, with its great achievements and conquests, with its distinguished place in the record of human thought, with its crystal-clear eternal beauty and its great importance in the life of mankind. A mathematician should rejoice at his good fortune in being acquainted with so delectable a subject upon which to expend his time, energy and love.

Hasse welcomes this occasion and indeed every opportunity, to spread broadcast an understanding of the true nature and essence of mathematics, to tell all the world, including those to whom mathematics is anathema, what mathematics is and what it means to him, and to show that mathematicians are really human beings. There are not many more fitted than he to do this. He is a real artist with his soul engrossed in mathematics, and he can draw upon an extensive musical and artistic knowledge for his illustrations and comparisons, and for revealing his whole-hearted devotion to mathematics.

The booklet contains three chapters. The first is on Mathematics as a Science. He recalls that the mathematical knowledge of the ancient Babylonians and Egyptians was founded on an empirical basis ; and that even now there are many people who consider that calculations and calculators are synonymous with mathematics and mathematicians. It was the ancient Greeks who gave a different turn to mathematical thought. They were not content with empirical rules, *e.g.* the method of setting out a right angle. They wished to put them on a sound logical basis and succeeded in doing so. Here one sees the inborn or natural urge of civilized mankind to seek the truth, a quest which finds its highest expression in mathematics. It was this desire which made Hasse resolve to become a mathematician.

The second chapter deals with Mathematics as an Art. There are many, especially women, who wonder why, or how, a mathematician can find satis-

faction in an activity that apparently offers nothing for the heart or soul. Hasse gives a personal account which shows that he sees, in ever increasing measure, in mathematics, all the characteristics of an art. This is taken from his knowledge and experience of music; and it is well known that many distinguished mathematicians have been musically inclined and that this has often given occasion to consider the inner relation between music and mathematics. He is convinced, however, that one could speak similarly in considering an art other than music. Hasse gives many instances, *e.g.* beauty in detail, in structure, and as a whole, both in presentation and in proof. These he illustrates with a determinant of the third order, showing the great aesthetic difference in the formula for its value according as one uses nine letters, $a, b, c, d, \ldots$ for the elements of the determinant or the now usual suffix notation a_{11}, etc. Another instance arises by showing how best to enunciate in the simplest way the theorem concerning the representation of a prime of the form $4n+1$ as the sum of two squares. These and the previously mentioned theorem of Pythagoras are the only mathematical theorems given in the lecture.

His comparisons between musical and mathematical style, structure and content, are well worth while. Mathematical writers will profit considerably if they pay heed to his admonitions that they should do all in their power to make their exposition simple, clear and attractive and in everyday language whenever possible.

The final chapter deals with the aspect of Mathematics as Power. Mathematicians have long been familiar with the power given by a new discovery; thus the invention of the calculus enabled them to resolve a host of problems of many kinds, on areas, tangents, maxima and minima, which previously required for each separate problem a distinct method often not easily found. Everyone knows of the great accomplishments in science and technology made possible by mathematics, and without which progress would often be impossible. But there is no need for the pure mathematician to justify his existence by these applications. In fact, it is often the purest and most abstract mathematics, really wonderful creations of human thought, that have been so important and fundamental in practical applications. Thus the theory of complex numbers, developed by Gauss in his number-theory investigations, plays a vital part in modern physics and in electrotechnics. The abstract algebra developed by the well-known Göttingen mathematician, Emmy Noether, supplied the essential stimulus to Heisenberg's conception of quantum mechanics, the foundation of atomic physics. The late war and modern industry show that the pure mathematician is not behind his applied colleague, but rather ahead when it comes to dealing with the manifold problems of every kind that arise.

Certainly Hasse's lecture will convert all his readers to his point of view, that mathematics and mathematicians are well worth while, and have deep human interest. He will make his readers realize that mathematics is still the Queen of the Sciences, commanding the loyalty and devotion of all her subjects, of whom he is proud to be one. No more pleasant tribute can be given her than this delightful, thoughtful, beautifully composed lecture.

L. J. MORDELL.

The Higher Arithmetic. By H. DAVENPORT. Pp. 172. 8s. 6d. 1952. (Hutchinson)

This book is an introductory account of the theory of numbers. It commences with a chapter on factorisation and the primes, dealing with the laws of arithmetic, the principle of proof by induction, the unique factorisation theorem, the euclidean algorithm and other properties of the primes,

including an interesting algorithm for factorisation recently discovered by Draim.

Chapter II on congruences contains the theorems of Fermat, Wilson and Lagrange together with a discussion on congruences in several unknowns. Quadratic residues are introduced in Chapter III and the theory is developed sufficiently far to be able to end by proving the law of quadratic reciprocity.

Chapter IV, the longest in the book, is concerned with continued fractions. The recurrence relations, Euler's rule, and the convergents of the infinite simple continued fraction are proved. It is also proved that any quadratic irrational number has a continued fraction which is periodic after a certain stage, and this naturally leads to a discussion of Pell's equation. A table containing the continued fraction expansion for $\sqrt{N}$ and the smallest integers x, y such that $|x^2 - Ny^2| = 1$ is given for $N \leqslant 50$.

Chapter V deals with the question of the representation of a number n as the sum of two, three or four squares and gives a number of constructions for x, y such that $x^2 + y^2 = p$ where p is a prime of the form $(4k + 1)$. Quadratic forms are dealt with in Chapter VI. The author restricts himself to binary forms, dealing in turn with the equivalence of such forms, the representation of integers by positive definite forms and the reduction of positive definite forms. A table of reduced positive definite forms of discriminant $-D \leqslant 83$ is given and Dirichlet's class-number formula is discussed.

The last chapter on Diophantine equations obtains the general solution of $x^2 + y^2 = z^2$ and discusses the solubility of (*a*) $ax^2 + by^2 = z^2$ and (*b*) $x^3 + y^3 = z^3 + w^3$. Fermat's problem and the Thue-Siegel theorem are also considered.

It is obvious that the work has benefited considerably from the author's researches in the many topics discussed. Each chapter ends with a set of references and notes discussing some of the more intricate and interesting points of the chapter. These notes should prove invaluable to the serious student of the theory of numbers and will also be useful to those who will have just met this theory for the first time and wish to pursue it in more detail. By his simple treatment and lucid exposition the author has been able to overcome the difficulties which are inherent in the subject itself and to produce a work of elegance and charm. P. M. H.

Vorlesungen über höhere Mathematik III. By A. DUSCHEK. Pp. ix, 512. 62s.; bound, 66s. 1953. (Springer, Vienna)

Dr. Duschek describes his third volume (reviews of the earlier volumes are in *Gazette*, XXXIV, 145; XXXV, 221) as a two-fold compromise, first between the rigour of the mathematician and the needs of the physicist or technician, secondly between the claims of the three main sections of this volume. As regards the first, he is likely to be accused in this country of making too many concessions to the standpoint of the pure mathematician, but at the heights now reached, a secure foothold is essential. The three sections deal with differential equations, ordinary and partial, with the calculus of variations, and with functions of a complex variable; all this in 460 pages (for hints and answers to the exercises take up 40 pages), with no very notable omissions, suggests that the second compromise has not failed.

The hundred pages on ordinary differential equations give a convenient blend of familiar methods of solution,* existence theorems, and applications;

* The Laplace transform is reserved for Vol. IV, but the reference to Heaviside's operational method on p. 108 is somewhat misleading, since it can be read as implying that there is no difference between the Heaviside method and the D method so familiar in English textbooks, a method much older than Heaviside, even older than Boole, to whom so much of its development and of its currency in English texts is due. A definitive account of the history of these processes is given by J. L. B. Cooper, *Gazette*, XXXVI, 5–19.

the space devoted to numerical methods of solution is, however, disappointingly small. The seventy pages on partial differential equations omit some of the so-called standard methods of solution, but the loss is not serious, since the author gives an excellent insight into the structure of the topic ; the subject is often a stumbling block to the ordinary student, and he would find these pages heavy going, but should emerge with a sound grasp of the main principles.

The third section, on complex variable, calls for little comment ; most of the work is classical, and there is small scope for variety of exposition. The striving after brevity shows its weak side in the accounts of the special functions, for the reader is hardly likely to appreciate the beauty or the utility of the elliptic functions (Weierstrassian and Jacobian) and the theta functions from the mass of formulae concentrated into forty pages.

In the second section, on the calculus of variations, the author is at his best. Here, it seems to me, is an account which would be equally valuable to the young pure mathematician, wanting a non-specialist's acquaintance with the main lines of the theory, and to the advanced technician, wanting the technique on a sound basis but unable to face the severe task of reading one of the classical treatises. Worked examples and exercises for the student illustrate all the main points. The exposition demands considerable attention when we reach tensors and generalised spaces, but if this section is mastered the resulting account of the Hamilton-Jacobi theory presents no difficulties in its crisp and lucid development.

If, unlike the earlier volumes, this cannot be read in an easy chair, that is because of the nature of the contents, not because of any falling-off in the author's care or competence. The whole book is well worth a place in the library, if a little beyond the purse of the ordinary student ; and I look forward with pleasure to the prospect of seeing Dr. Duschek's handling of the highly interesting topics with which the fourth and final volume will deal : boundary value problems, expansions, integral equations and the Laplace transform.

T. A. A. B.

Calculus of Variations by Parallel Displacement. By W. S. KIMBALL. Pp. viii, 543. 50s. 1952. (Butterworth)

If y is a differentiable function of x, $y=f(x)$, then the maxima and minima of y occur at points a such that $f'(a)=0$ and, if $f'(a)=0$, then y is a maximum if $f''(a)<0$, a minimum if $f''(a)>0$. Similar formulae hold for functions of several variables.

The calculus of variations is concerned with integrals

$$J=\int_{x_1}^{x_2} F(x, y, y')\,dx$$

taken along arcs in the (x, y)-plane ; thus the celebrated brachistochrone problem expresses the time T required by a particle starting with initial velocity v_1 to fall from (x_1, y_1) to (x_2, y_2) along a given arc by the formula

$$T=\frac{1}{\sqrt{(2g)}}\int_{x_1}^{x_2}\sqrt{\left\{\frac{1+y'^2}{y-\alpha}\right\}}\,dx, \quad \text{where } \alpha=y_1-(v_1^2/2g).$$

J is, in general, a function of the arc chosen, and the problem is to minimise J, that is to say, to find an arc E_{12} joining (x_1, y_1) to (x_2, y_2) such that $J(E_{12})\leqslant J(E)$ for all admissible arcs E. (The corresponding problem of maximising J is analogous mathematically but does not seem to arise so naturally in physical applications.) The author's object is to generalise the elementary differential calculus and express necessary and sufficient conditions for

$J(E_{12})$ to be a minimum in terms of "derivatives" of J with respect to changes in E; in the author's own words, "We approach the subject of the calculus of variations by what is essentially the same delta process that is used at the outset of calculus to derive formulas for derivatives and instantaneous rates of change."

Let us write

$$M = F - y'F_{y'}, \quad N = F_{y'};$$

then

$$\int F\,dx = \int M\,dx + \int N\,dy.$$

The author maintains that M, N are the rates of change of J due to changes in the direction of the path at (x, y); in his symbols,

$$\frac{\partial J}{\partial x} = \frac{\partial J}{\partial dx} = M, \quad \frac{\partial J}{\partial y} = \frac{\partial J}{\partial dy} = N. \qquad \text{.....(i)}$$

Let (x_i, y_i), (x_u, y_u) be points, in that order, of the path E (directed from (x_1, y_1) to (x_2, y_2)). Then the second rule of the delta process states that changes in J due to parallel displacement of the path between (x_i, y_i), (x_u, y_u) are represented by

$$\left.\begin{aligned} \frac{\partial J}{\partial x} &= M_i + \int_{x_i}^{x_u} F_x\,dx - M_u \\ \frac{\partial J}{\partial y} &= N_i + \int_{x_i}^{x_u} F_y\,dx - N_u \end{aligned}\right\}. \qquad \text{.....(ii)}$$

The author also defines an area derivative $dJ/d\sigma$ ("a derivative with respect to the area adjoining the integration path at (x, y)") and asserts the rule

$$\frac{dJ}{d\sigma} = \frac{d}{dy}\frac{\partial J}{\partial x} = -\frac{d}{dx}\frac{\partial J}{\partial y} = \frac{d}{dx}F_{y'} - F_y = N_x - M_y. \qquad \text{.....(iii)}$$

Returning to (i) the author notes that M, N may be considered as components of a vector $\mathbf{R} = M\mathbf{i} + N\mathbf{j}$. Then

$$\int F\,dx = \int (M\,dx + N\,dy) = \int \mathbf{R}\,.\,\mathbf{ds},$$

where, as usual, $\mathbf{ds}$ is the vector displacement $dx\,.\,\mathbf{i} + dy\,.\,\mathbf{j}$; $\mathbf{R}$ is called the vector integrand and its curl gives the partial derivatives of J by parallel displacement.

On the basis of his notion of a derivative of J, the author derives all the classical results of the theory. Thus the vanishing of the area derivative gives immediately the Euler differential equation for an extremal arc. The Weierstrass, Legendre and Jacobi conditions for a minimising arc are obtained, the theory of the Hilbert integral (based on Weierstrass's field of extremals) is developed, the envelope theorem is proved, the Weierstrass-Erdmann corner condition is derived, and in the final chapter it is claimed that the celebrated Weierstrass sufficiency condition is established.

It is not stated explicitly for what level of reader this book is intended; however, since the contents (apart from copious and most interesting applications) are broadly co-extensive with those of Bliss's Carus monograph, the reviewer has assumed that it is intended for non-specialists as well as for specialists. Since the reader is asked to accept and adopt new methods for the derivation of known results, it is natural to subject the book to the two tests: (*a*) are the new methods superior to the old, and (*b*) are they clearly

explained? It is the reviewer's opinion that a negative answer must be given to (*b*) and that (*a*) is incapable of precise application.

As a work of mathematics this book seems to suffer from serious defects. The first is the astonishing prolixity of style (it is perhaps significant that the author's degrees include that of Ll.B). This defect is particularly apparent in the introduction and first two chapters, just where precision and conciseness are so necessary. Two quotations must suffice: on p. 19, we read "Thus the derivatives 1.17 are area derivatives, in that the differentials of the independent variables are the unshaded parallelogram areas $\delta x\,dy$ and $\delta y\,dx$ of Figure 1.3, and when these are set equal to unity or divided out to appear in the denominators of the left members (which amount to the same thing), then there remains the change in J due to unit change for these differential areas adjoining the curve or rates of change of J with respect to the adjoining areas." The title of section 2 of chapter 2 reads: "The slope y' and differentials, including sign, involved in the vector integrand 2.8 are necessarily the slope and differentials, including sign, of the path of integration for all the calculus of variations integrals J of 1.1 and 2.9 determined by x, y and y' the slope of the path of integration."

The second serious defect is the looseness of notation. Thus for example no distinction is made between a vector and its absolute value and one gets even (p. 3) "... where R is the magnitude of a vector $R=iM+jN$ whose direction makes angle with the length element vector, $ds=\sqrt{(dx^2+dy^2)}$ and whose magnitude is $R=\sqrt{(M^2+N^2)}$ ..." Rule (i) above provides another example of bad notation, and in rule (iii) F_y is the partial derivative of F as a function of x, y, y' while M_y is the partial derivative of M when y' is a function of x and y.

A third fundamental defect lies in the author's use of loosely-defined limiting processes throughout his arguments. For example, we have (p. 15): "Then also $x_{u-1}=x_u-\Delta x_u$ approaches x_u and y_{u-1} approaches y_u, so that 1.10 becomes in the limit equations 1.11 (see overleaf) with an integral term, and with F of 1.2 replaced on the left by J_i^u, and the notation of 1.1, taken as the limit of F." The author asserts (p. 32): "It is also well known that a differential is a finite variable approaching a limit zero." The author admits his attitude in section V of the introduction, entitled "The emphasis is on geometry and mechanics rather than the foundations of analysis." He writes: "... The ϵ, δ process in connection with limits is never used. Various conclusions and theorems are stated and illustrated without attempting to set forth their all inclusive meets and bounds, according to the extremely exacting modern standards of mathematical rigour. It is left to the student to accept, curtail or extend the conclusions or theorems herein to the extent of his understanding, inclination and background guided by the usual interpretations given to mathematical symbols and clear statements in precise English ..."

Thus the arguments themselves are essentially plausibility arguments and the question of their superiority to the classical arguments does not really arise. Whether an applied mathematician would find the methods here described (in 531 pages) more congenial than those of Bliss, Carathéodory, and Hadamard it is difficult for the reviewer to say; but it would be less than fair to end this review without saying (*a*) that the reviewer is a pure mathematician who has made no semblance of an original contribution to the theory of the calculus of variations and (*b*) that the author has assembled as comprehensive a collection of applications as could be wished. It may well be that the book will continue to be valued for this collection even if the methods the author has been at such pains to recommend do not find general acceptance.

P. J. HILTON.

Abstract Set Theory. By A. A. FRAENKEL. Pp. xii, 479. 76s. 1953. (North-Holland Publishing Co., Amsterdam)

The new series " Studies in Logic and the Foundations of Mathematics " is to include three works on the theory of sets, of which the subject of the present review is the most elementary. The other two works are Fraenkel's *Foundations of set theory*, and *Axiomatization of set theory* by Bernays, Fraenkel and Borgers, neither of which has yet appeared.

In view of the forthcoming *Foundations* Fraenkel is able, without confining *Abstract set theory* to purely mathematical topics, to reserve critical consideration of the more philosophical and logical aspects of the subject for the second work. Nevertheless he has not been contented with a purely naïve development of set theory but has chosen a course midway between Cantor's total identification of set and class, and a strict axiomatic separation. The treatment may best be described as an informal introduction to the Zermelo-Fraenkel system. The contents of the book are roughly those of the first half of the author's well-known German text, but there are many points of difference, not only the distinction between class and set already noted, but also in regard to new material and new proofs.

Set theory is an exciting subject—perhaps no other branch of mathematics has more of the magic and wonder of the Fairy Story—and the present account has a subtle dramatic quality, compounded of delight and depth of comprehension, well suited to its subject. Even the experienced reader recaptures some of the thrill of first discovering the cardinal similarity of the line and the plane, the arbitrary small interval and the whole of space. Fraenkel is careful to emphasise the part which continuity plays in the rehabilitation of the concept of dimension, and gives a very simple proof that there is no continuous mapping of a plane on a line. Two proofs are given of the famous Schröder-Bernstein theorem (that $a \leqslant b$ and $b \leqslant a$ entails $a = b$, where a, b are transfinite cardinals), both simpler than that, for instance, in Littlewood's *Elements of the theory of real functions*, but neither as short and crisp as one in Sierpinski's recent *Algèbre des ensembles*.

Cantor's famous continuum hypothesis that the cardinal, aleph, of the continuum is equal to the cardinal, aleph-one, of the well-ordered series of ordinals of the second class (ordinals of denumerable well-ordered sets), is discussed in some detail. It is proved that aleph is not less than aleph-one and not equal to aleph-omega, and there is a reference (without details) to Gödel's 1938 result that the hypothesis cannot be refuted, that is, that the affirmation of the hypothesis does not introduce a contradiction into set theory (if there is not one there already).

Of work on transfinite ordinals too recent for inclusion in the book mention may be made of Denjoy's discovery of a constructive representation of each ordinal of the second class as the limit of a unique sequence of ordinals, which was announced at the Austrian Mathematical Congress in the autumn of 1952.

One of the novelties of the book is a discussion of definition by induction (including transfinite induction). Following Dedekind, Fraenkel holds that the familiar definition of addition by induction,

$$m + 0 = m,\ m + Sn = S(m + n) \quad \text{.....(i)}$$

(where Sx denotes the number after x) is inadequate. For, it is argued, we have not defined $m + n$, with a variable n, only $m + \mathrm{p}$ for any numeral p. Dedekind proved the proposition

" for all n, x there is a g such that for all y

$$y < n \rightarrow g(0) = x \,\&\, g(Sy) = S(g(y)) \text{ } "$$

by induction over n, whence taking n to be Sy, it follows that for any x, there is a g such that for all y

$$g(0) = x \,\&\, g(Sy) = S(g(y))$$

and from this, (in a suitably designed formalism) we derive the existence of a function $\Sigma(x, y)$, say, which satisfies the conditions $\Sigma(m, 0) = m$, $\Sigma(m, Sn) = S\Sigma(m, n)$. Landau, in the *Grundlagen der Analysis*, uses a neat device of Kalmar's to avoid introducing the inequality $y < n$ in the proof (since it is far simpler to define "$y < n$", by means of addition, as "there is an x such that $n = y + Sx$", than by means of Dedekind chains) and derives the existence of the sum function directly from the proposition

"for all x there is a g such that, for all y

$$g(0) = x \,\&\, g(Sy) = S(g(y))",$$

which is proved by induction over x. Interesting though this work of Dedekind and Kalmar is, it is quite wrong to conclude that the definition of addition by means of the equation (i) is in any need of support. It is entirely a question of the type of formalism in which arithmetic is being set up. Dedekind (and Landau) are thinking of a system in which there is an existence operator for function signs (the second order predicate calculus) and any particular function (which satisfies suitable conditions, like those for addition) is definable explicitly, which means that the function sign of the particular function is just an abbreviation for some other expression and may be eliminated at will. But in a system like recursive arithmetic, there are no operators over function signs and the sum function $x + y$ is introduced by means of the axioms

$$x + 0 = x, \quad x + Sy = S(x + y).$$

These axioms serve to determine the sum of any two natural numbers $\mathfrak{p}$ and $\mathfrak{q}$, and in fact by means of them we can eliminate the sign " + " from any expression "$x + \mathfrak{p}$" with a variable x and definite numeral $\mathfrak{p}$. But we cannot eliminate the sign in the context "$x + y$" with variables x and y, so that the axioms do *not* constitute an *explicit* definition of addition. None the less they define addition as completely as any definition can for by their means the sum of any two natural numbers is determined. In one system definite functions are defined by means of function variables, in another they are subject to appropriate axioms.

The distinction between class and set which (following Zermelo) Fraenkel establishes on a basis of seven *principles* (informal axioms) is of course a difficult one for a beginner to grasp and in the formulation of some of these principles the author's happy gift for exposition has been a little less in evidence than elsewhere. The whole point is that some classes are honoured with the title "set" and others are not, but to say that "if a and b are different objects there *exists* the pair $\{a, b\}$" (my italics) can only lead the uninitiated into despair. Existence is not in question; the principle merely affirms that pairs are honoured classes and *called* sets. A similar point of wording arises in connection with sum-set, and, less obviously, in the principle of infinity.

The last 132 pages of the book comprise a superb bibliography prepared with immense care and scholarship, with numerous cross references, covering a very wide field of study in set theory, mathematical logic and the foundations of mathematics, and listing books, articles and, in some cases of special importance, also reviews. In one or two cases the dates of first editions have unfortunately been omitted.

The printing, the paper and lay-out of the book are of an exceptionally high quality, making the book very easy to read and handle, but also raising the price rather high for the third-year student in mathematics or philosophy for whom the book is designed.

R. L. GOODSTEIN.

Noordhoff's Wiskundige Tafels in 5 Decimalen. 5de druk van Tafel H. By P. WIJDENES. Pp. viii, 269. 8.75 Dutch florins. 1953. (P. Noordhoff, Groningen)

This is one of several tables published by Noordhoff and edited, alone or jointly, by Wijdenes. The small passages of text are mostly in six languages; the English title is *Mathematical Tables to 5 Decimal Places*. Most of the volume is occupied by logarithmic and trigonometrical tables, the ordinary sexagesimal units of angle being used. The angular interval is normally 1′, but is reduced to 10″ or 1″ in the case of some functions of small arguments. There are other elementary tables, and the volume concludes with several brief tables of higher functions (factorial function, exponential integral, sine and cosine integrals, error integral, Bessel functions). There appears to be no information which cannot fairly easily be found elsewhere, but the collection as a whole may suit some of those for whom it is intended, namely "students and laboratory workers in universities and technical or service colleges". The book is excellently printed and bound; users will find it convenient that Tables II–III and V are distinguished by being printed on shell pink and pale green paper respectively. A. F.

Table of the Reciprocal of the Gamma Function for Complex Argument. By J. P. STANLEY and M. V. WILKES. Pp. 103. 36s. 1953. (Geoffrey Cumberlege, London)

This table first appeared in 1950 as a publication of the Computation Centre, University of Toronto, and is now published unchanged in this country. It gives to six decimal places the real and imaginary parts of $1/\Gamma(x+iy)$ in the region $-\frac{1}{2} \leqslant x \leqslant \frac{1}{2}$, $0 \leqslant y \leqslant 1$, at interval 0·01 in both x and y. The values were calculated on the electronic digital computer EDSAC at the University Mathematical Laboratory, Cambridge, of which Dr. Wilkes is Director, during a period of research at Cambridge by the Toronto author. The values were checked by differencing in both directions on a National accounting machine, and are believed to be correct within 0·7 units of the sixth decimal place. A loose sheet gives about forty corrections, partly of errors which occurred in the preparation for press and partly of illegible entries. This list of corrections has also appeared in *Mathematical Tables and other Aids to Computation*, 5, 160, 1951.

Previous tables of the gamma function of complex argument are not numerous, and have usually covered a larger region with a coarser lattice (for example, with intervals 0·1 in x and y). Such computations have been valuable as a basis for maps in the complex plane, like the map of $1/\Gamma(z)$ in the well-known tables of Jahnke and Emde. The present tables cover a restricted region at a fine interval; the computer who requires values outside the range of the table is referred to the standard formulae for $\Gamma(z+1)$ and $\Gamma(2z)$.

The contents of the table are important, and as a first fruit of the Edsac the publication has historical significance. Most computers would be glad to possess it. One wishes, however, that it could have been printed from type, and consequently in handier format. For a publication in machine script of variable quality (medium to poor), the price will appear very high in this country, though it is a more or less reasonable equivalent of the transatlantic price of 4.50 dollars. A. FLETCHER.

The Planets. Their Origin and Development. By HAROLD C. UREY. Pp. xviii, 245, with 16 figs. and one folding plate. 30s. 1952. (London: Geoffrey Cumberlege, Oxford University Press)

The Silliman Memorial Lectures of Yale University give, by direction, especial prominence to astronomy, chemistry, geology and anatomy. This

thirtieth volume of the Lectures combines aspects of the first three disciplines in a fashion more direct than any of its predecessors in a long and distinguished series. Professor Urey is a physical chemist, and after all, the origin and development of the planets are problems as much in physical chemistry as in astronomy ; it need therefore occasion no surprise that his incursion into a field all too long regarded as the special province of the astronomers is fascinating and thought-provoking. A book of this kind is all the more welcome now that a new generation of geophysicists is questioning much that appears on the first page of every school geography text, and young astrophysicists are adding to existing theories of the origin of the solar system at a greater rate than observers are adding to the number of bodies in it.

Professor Urey approaches the problem, as he says, " the hard way ", through a study of the Moon's surface, the chemical composition of the Earth and Mars, the physical structure of the Earth, and so on. Marshalling as he goes along a bewildering variety of facts and deductions concerning such apparently unrelated topics as the heat balance of the Moon, the oblateness of the terrestrial planets, the photo-chemistry of atmospheric processes, the formation of folded mountains, the secular accelerations of the Sun and Moon and a dozen other subjects, he gradually builds up a picture, no less convincing than any other, of the development of the solar system. This theory falls in the " non-catastrophic " class and involves a necessarily complicated schedule of condensations from an initial dust cloud such as can be observed elsewhere in the Galaxy to this day. The main differences from current theory are his suggestions that the Earth and other inner planets were formed at much lower temperatures than previously believed (indeed the Earth, as a planet, has never been hot), and that iron was formerly distributed more uniformly than now in the silicate phases of the Earth, *i.e.* that the iron core is of relatively recent formation.

It is not to be expected that a book of this sort will provide the last word—even the author's last word—on such a subject. It is sufficient that it should throw light on places previously dark and raise questions on neglected aspects of the subject. From these points of view it is highly successful ; no future workers in the many fields covered can afford to neglect its arguments and conclusions.

A. H.

Sir James Jeans. A Biography by the late **E. A. Milne.** With a Memoir by S. C. Roberts. Pp. xiv, 176, with frontispiece, 1 plate and 7 figs. 21s. 1952. (Cambridge University Press)

The first half of this book is conventional biography, conventionally written ; the second six chapters assess Jeans's status as a man of science and set into their twentieth-century background his contributions to the kinetic theory of gases, to electromagnetism, to cosmogony and to philosophy. That the author is obviously happier in the second half will come as no surprise to those who knew Milne ; and indeed, paradoxically enough, a better idea of Jeans's personality emerges from the account of him as a scientist than from the somewhat pedestrian description of his personal life in the early chapters. For a picture of the man in the clearest focus the reader must turn to the short memoir by S. C. Roberts, sometime Secretary of the Cambridge University Press, which opens the book. Sketched here in a few pages is the man himself, wart and all.

It follows that to the great majority of Jeans's popular audience, which must have run into millions, this book will prove something of a disappointment. But to his scientific audience, here is his work evaluated brilliantly, his controversies with Eddington analysed with lucidity (and none the less so because Milne himself was so closely concerned), and the effect of his philo-

sophy on current scientific thought described with sympathy. If Milne himself takes sides on occasion (the viewpoint of kinematic relativity is not neglected) one would not have it otherwise.

The author died before he could give the book a final revision. This was left to S. C. Roberts as regards the biographical chapters and to G. J. Whitrow for the rest. No lack of integration appears, and the book as a whole is produced as impeccably as the publishers' imprint would suggest. A. H.

Cosmology. By H. BONDI. Pp. 179, with 6 figs. 22s. 6d. 1952. (Cambridge University Press)

The subject of this book has had a chequered history in the evolution of human thought. For long regarded as a branch of theology or of metaphysics, the study of cosmology has gone through a brief period in which it was claimed by general relativity theory and seems now to be emerging as a branch of physics which must survive observational tests just as any other branch. Mr. Bondi firmly denies that cosmology is to be thought of as a minor branch of philosophy or logic and claims that he takes his stand with the growing band of physical cosmologists.

In this book he first of all summarises the observational evidence, in itself no light task where measurements of such delicacy are concerned. He then describes the cosmologies of Newtonian dynamics, of general relativity and of kinematical relativity before enunciating the " perfect cosmological principle " behind the steady-state theory which he first put forward with Gold in 1948. As is known, this leads to much the same result as Hoyle almost simultaneously deduced from a modification of the field equations of general relativity, namely, the necessity for continuous creation of matter. A final chapter summarises the present position of the various theories and indicates how future observations may best winnow the wheat from the chaff.

Mr. Bondi writes persuasively. His description of observational cosmology is more accurate than might be thought from some of his statements (*e.g.* that the proper motion of the " fixed " stars was discovered after Mach published his hypothesis—in fact it was 175 years before); and his review of the theoretical side of the subject will be welcomed by many who had resigned themselves to drowning in the flood of recent cosmologies, New or simply new. That the approach he favours particularly will not meet with universal approval (for instance, Professor H. Dingle in his 1953 Presidential Address to the Royal Astronomical Society stigmatised it scathingly as the very contrary of science) need not deter anyone from reading this well-written " progress report " from cover to cover. A. H.

General Topology. By W. SIERPINSKI. Translated by C. C. Krieger. Pp. xii, 290. 48s. 1952. Mathematical expositions, 7. (University of Toronto Press; Geoffrey Cumberlege, London)

This book, completed in 1948 and translated in 1952, differs from the author's *Introduction to General Topology*, (Toronto, 1934) in the axiomatic treatment, in the scope of the subject matter, and in the number of examples provided. However, it is not likely to establish itself in this country as a standard introduction to point-set topology, a role played satisfactorily by Kuratowski's *Topologie* (in French) and Newman's *Topology of Plane Sets of Points*. Rather its value will be as a work of reference and a source-book of examples of spaces with specific properties (and therefore as a source of counter-examples of conjectures). Thus, for example, it will be very useful to have readily available a non-metrizable Hausdorff space with a countable basis, or two non-homeomorphic subsets of a metric space, each a (1, 1) continuous image of the other.

The treatment in this book is somewhat unsuitable for the English reader, because he is assumed at the outset to be totally unfamiliar with the elements of topology but fairly conversant with the Cantor theory of transfinite arithmetic. This is rarely the situation the recent graduate in this country finds himself in. It is true that an appendix giving the elements of the Cantor theory is provided, but it is highly condensed and does not affect the fact that the elementary topological theorems in the early chapters are interspersed with rather recondite results on the cardinal number of certain aggregates and unfamiliar transfinite constructions. The definitions themselves are affected by the author's point of view; thus the concept of compactness is based on the notion of m-compactness (m being an arbitrary cardinal), and even then one finds that "compact", in its usual sense, is called in this work "compact-in-itself".

The book is not very well edited and there are indications that the translator may not have been very familiar with the mathematical theory. The symbols $K - H$ (K, H being sets), ϕ, $\subset$ are used without explanation, and the notion of a bounded set is nowhere clarified. Borel sets are discussed anonymously on pp. 167 *et seq.*, but not named till p. 205 of the next chapter. In the appendix, the second sentence of §5, p. 264, is false if "on" has its usual meaning and formula (26), p. 268, is false unless E is countable. On p. 33, line 20, "$\geqslant \overline{\overline{E}}$" should be "$> \overline{\overline{E}}$" (incidentally, it *is* confusing, despite footnote 15, to use $\overline{E}$ for the closure of E and for the cardinal of E). On p. 109, line 19, "E" should be "E_1", and on line 21, "$S(a, \epsilon)$" should be "$S(a, \frac{1}{2}\epsilon)$. On p. 125 "$\left[\frac{nx_k}{n}\right]$" should be $\frac{\text{“}[nx_k]\text{”}}{n}$ throughout. On p. 163, last line, "$q \epsilon T$" should be "$q \epsilon E$". On p. 166, line 29, "Γ" should be "Γ_n", and on p. 167, line 27, "Q_γ" should be "G_γ". In theorem 91, p. 185, the second appearance of the letter E should be expunged.

It is curious that this work opens with a discussion of Frechet (V)-spaces, which are so general as to present almost wantonly curious features, and then defines a topological space in Chapter II so restrictively that the global topology (the "strongest" or "coarsest" topology in which the whole space and the empty set are the only open sets) is excluded; in fact, the T_1-separation axiom is assumed from the outset. However, the chapters on metric and complete spaces, which form the main part of the book, deal very comprehensively with their subject matter, and earn this work its place as a reference work for information on these topics. P. J. HILTON

Foundations of Algebraic Topology. By S. EILENBERG and N. STEENROD. Pp. xv, 328. 48s. 1952. Princeton Mathematical Series, 15. (Princeton University Press; Geoffrey Cumberlege, London)

Combinatorial homology theory is based on the idea of a triangulation of a given topological space. That is to say, the space is covered by a set of non-intersecting subspaces of a special kind, called simplexes, and the set of all simplexes is called a complex, say K. The simplexes are oriented and those of a given dimension n generate the (free abelian) chain group, C_n. Incidence relations between simplexes of consecutive dimensions lead to boundary homomorphisms $\partial : C_n \to C_{n-1}$, and the kernel of ∂, Z_n, is called the group of n-cycles. All boundaries are cycles, *i.e.*, $\partial C_{n+1} \subset Z_n$, so that we may define the factor group $H_n = Z_n / \partial C_{n+1}$. Then H_n is called the nth homology group of K. The notion of homology group may be "relativized", so that $H_n(K, L)$ may be defined, where L is a subcomplex of K (roughly speaking, we obtain $H_n(K, L)$ by calculating modulo L); also a dual theory, leading to the cohomology groups, may be developed.

Early work in homology theory centred round the question of the topological invariance of the homology group; it was sought to prove that these groups depended on the underlying space only and not on the particular complex K covering it. This was proved by Alexander using the technique of simplicial approximation and the notion of the homomorphisms of homology groups induced by continuous maps of spaces. An excellent account of this proof is given by Pontrjagin.* A similar argument† defines the so-called singular homology groups of a space in a topologically invariant manner and then establishes the isomorphism of the singular and combinatorial theories.

In the present work the authors start off from an axiomatic basis. Suppose that, to each triple (X, A, q) consisting of a topological space X, a subspace A and an integer q, there is assigned an abelian group $H_q(X, A)$ and to each continuous map $f: X, A \to Y, B$ there is assigned a set of homomorphisms $f_*: H_q(X, A) \to H_q(Y, B)$. Finally suppose that there is also given a set of homomorphisms $\partial: H_q(X, A) \to H_{q-1}(A)$. Then this system of groups and homomorphisms is said to constitute a homology theory if seven axioms are verified. It has long been known that the combinatorial homology theory of triangulable spaces verifies the axioms; in Chapter III it is shown that any homology theory (in the axiomatic sense) defined on the set of triangulable pairs of spaces (X, A) is unique up to isomorphism, and coincides with the homology theory derived by the classical algorithm. An essential step in the proof is the characterisation of the chain groups in terms of the postulated homology groups of the axiomatic system.

Chapters V–VII are devoted to the construction of homology theories. Since it is required to define a homology theory based on the purely algebraic concept of a "chain complex", the constituents of the axiomatic homology must be generalised. This is effected in Chapter IV by the introduction of the notions of category and functor. This chapter is difficult on first reading, but the wide application of these notions renders a familiarity with them advisable. Chapter VII itself is devoted to an exposition of the singular theory and the isomorphism between it and the combinatorial theory is explicitly described.

The Cech homology groups are defined and discussed in Chapters IX and X. The axioms are verified for these groups: all except the "Exactness" axiom are valid in general, and this last is valid on the set of compact pairs (X, A). Moreover, the Cech groups are shown to be categorical for compact pairs if a further axiom, the so-called continuity axiom, is added. The continuity axiom is a natural requirement in the light of the definitions of the Cech groups as limit groups, and Chapter VIII is devoted to a description of the theory of direct and inverse systems necessary for the development of the Cech theory. The last chapter (XI) is devoted to applications to euclidean spaces (for example, the Brouwer fixed-point theorem).

The book is in the nature of a landmark in homology theory and the authors are to be congratulated for presenting such a complete and systematic theory in just over 300 pages. This is, however, not a book for beginners; for although it is logically self-contained and comprises most of the classical homology theory, its sophisticated approach does not always leave the reader quite clear as to just what the authors are setting out to prove. As indicated, the type of argument used, involving arrows going in several directions (but why not from left to right in the exact sequence?) is still somewhat unfamiliar, but the general suppression of the intuitively evident in favour of the precise

* *Introduction to combinatorial topology* (Graylock Press, Rochester, N.Y.).

† H. Seifert and W. Threlfall, *Lehrbuch der Topologie* (Teubner, Leipzig; Chelsea Publishing Co., New York).

renders the effort of reading somewhat arduous. The authors have provided an admirable preface and each chapter opens with an explanation of the nature of its contents. However, within the chapters themselves, the reader may be submerged beneath the weight of theorems and lemmas and find it difficult to relate what he is reading to the general context. This difficulty, however, is by no means wholly the responsibility of the authors ; we repeat that the material and techniques are new. Each chapter ends with notes (which are really additional material) and exercises, whose importance may well be held to justify their difficulty.

The proof reading does not come up to the high standard of the other aspects of the book. For example :

p. 143, 1.6 ; hypothesis (i) appears without relatum:
1.13 ; $\phi((hb)\times g)$ should be $\phi''((hb)\times g)$:
p. 146, 1.11 ; 9.16 should be 9.15 :
p. 173, 1.7 ; " continuous " should be " contiguous " :
p. 210, 1.8 ; $\prod_{\alpha} H^q(X, G)$ should be $\prod_{\alpha} H^q(X_\alpha, G)$;

p. 216, 1.27 ; $\pi_\alpha{}^\beta(x_\beta)$ should be $\pi_\gamma{}^\beta(x_\beta)$:
p. 217, l. 11, 18 ; 3.4 should be 3.5.

Also, on p. 177, 1.16, the symbol s seems to be used both for a simplex of K and for its barycentre (" Since s is a vertex of $Sd\,s$ and $Sd\,s \subset St\,s$, ... "). These small blemishes are mentioned only because of the importance of complete accuracy in a work of this nature and significance. P. J. H.

The Theory and Applications of Harmonic Integrals. By W. V. D. HODGE. 2nd edition. Pp. x, 282. 27s. 6d. 1952. (Cambridge University Press)

This is the second edition of the work first published in 1941, containing the author's theory of harmonic forms and integrals and their applications to algebraic varieties and the Betti numbers of the classical groups. The most substantial change consists in a revision of the proof of the fundamental existence theorem for harmonic integrals (Chapter III, sections 30–33). A bibliography of relevant papers published since 1941 has also been added, supplementing the references at the end of each chapter, but there is no index. P. J. H.

Typical Means. By K. CHANDRASEKHARAN and S. MINAKSHISUNDARAM. Pp. x, 139. 35s. 1952. (Tata Institute, Bombay ; London, Geoffrey Cumberlege)

As the first mathematical monograph of a new series of publications by the Tata Institute this book, intended primarily for use by professional mathematicians actively engaged on research, sets a high standard by its accuracy and clear presentation.

The scope of many convergence arguments in analysis can often be extended by an appropriate interpretation of any divergent series (or integrals) that occur. Each such interpretation gives rise to a method of summability. Although there are many inter-relations between different methods of summability it is usually the case that there is one method, or one type of method, that is of outstanding importance in any particular problem. Typical Means or Riesz Means are a method devised to handle the convergence problems that arise in connection with Dirichlet Series.

The first two chapters of this monograph are devoted to the definitions and fundamental properties of typical means. The third and longest chapter deals with their application to Dirichlet Series and the fourth chapter with the method of spherical summability applied to multiple Fourier Series.

The book is written with great care and is full of results which have previously only been available in widely scattered articles in various periodicals. Some results given here are new and all the theorems are proved with as much generality as is possible. Throughout the book there is a coherence of method and of the structure of the results obtained that enable the reader to grasp the main features of the theory comparatively readily. But it is in the nature of the subject that such a book as this cannot be easy to read.

The authors are to be congratulated on having performed a necessary and useful service in producing a book which fills a very obvious gap in mathematical literature. The information contained in this monograph is completely up-to-date and there is no doubt that it will serve as the starting point for many future researches.

At the conclusion of each chapter are "Notes" which refer to original papers and contain other relevant comments. The collection of these notes at the end of each chapter is not so convenient for the reader as their insertion at the proper place in the text would have been. It may be remarked from these notes that this is a branch of mathematics in which mathematicians in Britain have made fundamental contributions and that the subject owes a great deal to the work of G. H. Hardy, L. S. Bosanquet and B. Kuttner.

H. G. EGGLESTON

An Introduction to Plane Projective Geometry. By E. J. HOPKINS and J. S. HAILS. Pp. vii, 276. 27s. 6d. 1953. (Geoffrey Cumberlege, Oxford University Press)

This book is intended for First-year University students, or for those at Scholarship level. It develops projection on a non-metrical basis, and distance and angle are banished from the realm of fundamental concepts in the early chapters, reappearing later as cross-ratios involving the absolute points. During their exile, projectivity and involution hold complete sway. The conic is introduced as the product of two co-basal ranges or pencils, and projective relations between its points and lines are exhaustively discussed. The treatment is synthetic or analytic, whichever is more adapted to the purpose. The synthetic treatment could be improved in places by recasting an argument such as that on page 162: "$A=B$ and $C=D$. But $D=E$ and $A=C$. Hence $B=E$" in the form "$B=A=C=D=E$. Hence $B=E$," which is more economical, and much easier to follow.

Worked examples in the text are not numerous, and many of those chosen are of a dull, numerical texture, and do not illustrate the beauty and method of the subject. Some of the exercises for the student could have been used with advantage for this purpose. The distinction between a complete quadrilateral and quadrangle is stated, but is not always observed, and a student reading paragraph 14.33 can be forgiven for thinking that there is really no distinction, for a quadrangle is there referred to as a quadrilateral, with a peculiar property of "oppositeness" relating its sides. Of its "diagonal triangle" two vertices are vertices of the "quadrilateral" in spite of a previous warning against this very error. It is unfortunate, too, that the symbol $(PQRS)$ should be defined as $PQ.RS/PS.RQ$ on page 20, and as $PR.QS/PS.QR$ on page 69.

Some of the bookwork is obscure, even to one familiar with the subject. Para. 9.81, for example, seems to imply that conditions necessary to determine a conic completely are limited to those derived from its passing through given points. Not much light is thrown by the explanatory statement "the condition that a conic pass through a point is linear, for a conic cannot pass through a point more than once, whereas the condition that a line be a tangent

to a point-conic is not linear, for we imply here that two points of intersection of a line with a conic have moved up to coincidence." It is a surprise to read later (12.2) that "if a line is to be a tangent to a point-conic, two linear conditions are applied." Obscure, too, is the summary of 10.31: "If we choose three points of a conic to be the vertices of the triangle of reference, then the conic belongs to a family of conics (including *three* degenerate conics) which satisfy the same conditions." This same paragraph contains also the apparent justification of the statement "Of the family of conics passing through the vertices of a triangle there are three degenerate conics, each consisting of a line-pair." Theorem 3.62, ". . . any given line on a plane can be projected to infinity, and at the same time any two given angles in the plane can be projected into angles of given magnitude," has an infinity of exceptions, which should at least be mentioned. Theorem 11.92 contains the word "perspective", which should be "projective". The inaccurate statement is nevertheless proved, by the introduction of a false statement. The true statement of the theorem is implied in the corollary 11.921, where a correct conclusion is drawn by an improper application of the false theorem!

It is also but fair to warn the student against too frequent an application of the result in 15.73: "if four points are on a rectangular hyperbola, then any one of them is the orthocentre of the triangle formed by the remaining points."

These blemishes are unfortunate in a book which is excellently printed, with clear diagrams, and which contains a large selection of examples suitable for solution by projective methods. W. J. H.

The Mathematical Theory of Non-uniform Gases. An account of the kinetic theory of viscosity, thermal conduction, and diffusion in gases. By SYDNEY CHAPMAN and T. G. COWLING. 2nd Edn., pp. ix, 431. 60s. 1952. (Cambridge University Press)

The first edition of this book appeared in 1939, and has been out of print for some time; it is most gratifying that the authors have made available once again in a second edition this now classical treatise on gases. A decade and a half have passed since the book was first published, and a brief description of the contents of the book may not be out of place.

The kinetic theory of gases was founded by Clausius, Maxwell and Boltzmann and is concerned with atomic transport phenomena in almost perfect gases. Their work was continued and completed by Chapman and Enskog and in this volume an account of the subject is presented which for its authority, perfection and elegance of the mathematics is unlikely to be excelled for a long time to come. The mathematical presentation involves the use of a compact vector and tensor notation largely developed by Chapman and the late E. A. Milne; once mastered the notation will appear to be most successful and almost essential.

The first six chapters of the book are concerned with a description of the properties of a gas, and the derivation of the equations of transfer of Maxwell and the Boltzmann integro-differential equation; this is followed by a deduction of the properties of a gas in equilibrium. The main body of the book (chapters 7–15) contains the complete mathematical theory of non-uniform gases involving the solution of the Boltzmann integro-differential equation, and thence the derivation of the coefficients of diffusion, thermal diffusion, viscosity and thermal conductivity. The exact results differ from those obtained by approximate methods based on the idea of a mean free path and show that these may be in error by a factor of about 2, though they are correct as regards orders of magnitude. The next two chapters

(16, 17) are concerned with dense gases and the modifications which are introduced by Quantum Theory. In Chapter 18 the properties of ionized gases in the presence of electro-magnetic fields are discussed and the electrical and thermal conductivities of such gases are derived, for the first time with the same rigour and completeness to be found in the rest of the book.

The new edition differs little from the original—sufficient proof of the authority and completeness of this book. The new material is almost all concentrated in notes which amount to thirty-six pages and are concerned mainly with recent development. Minor alterations have been made in the text (for instance on pp. 189–90, 257–8, 272, 286, 320) and errors corrected. One of the most important additions is a table of values of a new factor relating to thermal diffusion of various mixtures.

The additional notes are available separately at the modest cost of 5s.

V. C. A. F.

The Theory of Homogeneous Turbulence. By G. K. BATCHELOR. Pp. xi, 197. 25s. 1953. (Cambridge University Press)

This is the first of a series of Cambridge Monographs on Mechanics and Applied Mathematics. It is an admirable exposition and a complete summary of all the work done in Homogeneous Turbulence up to the present day by an author who is one of the leading theoretical and experimental workers in the field. Apart from its direct interest to students of hydrodynamics it will be of especial interest to students of meteorology, aeronautics and other applied hydrodynamical fields where turbulence is an uncomfortable reality. Not only is this the first book of a new series but also the first attempt that has been made in this field to condense, clarify and make consistent a vast amount of work done during the past twenty years. The editors will do well if they can maintain the high standard set by this monograph.

Since Sir Geoffrey Taylor's paper in 1935 on Statistical Theory of Turbulence the subject of Turbulence acquired a new direction and the subsequent growth of the subject has been quite phenomenal. The aim of the theory at present and of the book is to explore and explain the simplest aspect of the turbulence problem, namely homogeneous turbulence in incompressible fluids, that is turbulence in which the average properties are the same at all points of the fluid medium. In order to make the problem tractable the assumption of isotropy at any point is made, so that the average conditions at a point are independent of direction at the point. These hypotheses are not entirely divorced from reality since these conditions are approximately true in the turbulence downstream of a square grid in a wind tunnel. The hypotheses are not valid generally in fluid turbulence, for example in large-scale atmospheric turbulence there is neither homogeneity nor isotropy, but it is evident that the establishment of an adequate theory of turbulence can best be done by attacking the problem first in its simplest form. The principal feature of the theory is the development of the equation for the correlation tensor $R_{ij}(\mathbf{r})$ between the contemporaneous velocities at two points separated by the vector $\mathbf{r}$. This equation does not depend upon the isotropy assumption but contains the tensor which derives from the non-linear terms in the original Navier-Stokes equations. It is the presence of this particular tensor, as indeed it is the presence of the non-linear terms in the original Navier-Stokes equations, which is the core of the difficulty of further development. Consequently the theory develops at this point partly empirically and partly on physical intuition. At this stage in the book we are introduced to the Kolmogoroff hypothesis which applies to the small wave end of the spectrum where the supply of energy from the lower wave numbers is balanced by viscous decay. Another chapter concerns itself with the large wave end of the spectrum

wherein lies the most considerable part of the energy and here we are given the experimental results concerning the decay of these large wave components together with a discussion of Heisenberg's form of the energy spectrum. The middle of the wave spectrum which is the portion principally responsible for the energy transfer appears to be the most difficult to deal with although it is quite clear that it is the non-linear terms which are responsible for the transfer. It is evident that the subject still bristles with major problems but it is of the utmost value to have had these central problems stated so clearly and to have at hand the experimental results on which future developments will be based. T. V. D.

Les systèmes axiomatiques de la théorie des ensembles. By HAO WANG and R. MCNAUGHTON. Pp. 54. 750 fr. 1953. Collection de logique mathématique (Gauthier-Villars, Paris)

This is an attractively presented brief summary of the several axiomatic foundations of set theory which have appeared over the past fifty years.

Russell's theory of types has of course pride of place. This theory (the so-called simple type theory) is based on three axioms, the first of which says that a class is completely determined by its members, the second that to every property of objects of type $\leqslant i$ corresponds a class of type $i+1$, and the third that there are infinitely many objects of type 1. Type theory is in many respects a "natural" formulation of the theory of sets but it suffers from the defect that there are important parts of mathematics which are only derivable in type theory if unpredicative definitions are allowed, that is to say, definitions of a class by means of a totality of which the class in question forms a part (the so-called vicious circle definition). Quite recently, however, in his Princeton doctoral dissertation, J. G. Kemeny showed that the consistency of type theory can be established in the Zermelo set theory, so that we may have as much confidence in type theory as in the rather more constructive system of Zermelo.

The essential difference between a set theory like Zermelo's, and type theory, is that in the former, instead of classifying objects in types, a collection belonging to a higher type than its members, and banning the formation of classes of mixed type, every class is admitted but a separation is drawn between classes and sets, every property determining a class but determining a set only in specified circumstances; for instance, a subclass of a set, determined by picking out the elements of a set which have a certain property, is a set. (The "Interpretation" given of this axiom, Z 3, p. 16, is rather unfortunately worded, giving the impression that *any* set x contains *all* the x's with the property $F(x)$).

Zermelo's system was extended by Fraenkel who added the axiom that a class which is (1, 1) correlated with a set is itself a set.

A more recent theory is that developed by Bernays in the *Journal of Symbolic Logic* between 1937 and 1948. This system differs from the Zermelo-Fraenkel theory by the introduction of class variables (in addition, that is, to the set variables). The account of Bernays' system concludes with a simple proof that the system is free from contradiction provided that the Zermelo-Fraenkel theory is so free. The proof utilises the Skölem-Löwenheim theorem that every consistent system has a model in the domain of natural numbers.

The next system outlined is due to Quine; in this system we meet again a distinction between classes and sets, but the distinction is here achieved by restricting sets to those properties which admit a potential typing known as stratification.

The account concludes with the description of a sequence of systems T_n of increasing type of which the first is the part of the Zermelo system which

refers only to finite sets. Reference is made to Tarski's formulation in T_{n+1} of a truth definition for T_n whence it is concluded that T_{n+1} admits a proof of freedom from contradiction of T_n. As examples of the applications of these systems mention is made of McNaughton's proof that it is possible to formulate in T_2 a demonstration that if the Zermelo-Fraenkel system is free from contradiction then so is the Bernays system, and likewise possible to formalise in T_3 a demonstration of the corresponding result for Quine's system and a comparable extension of it. R. L. GOODSTEIN.

Methods of Statistical Analysis. By C. H. GOULDEN. Second Edition. Pp. vii, 467. 60s. 1952. (John Wiley, New York ; Chapman and Hall)

This is more than just a second edition of a book which rightly became popular on its first appearance in 1939. Thirteen years have elapsed during which the subject of statistics has become ever more popular, and has spread its boundaries to an extent which makes it difficult to cover everything in a general text-book. It is a tribute to this book, and to its author, that the general plan has required so little change. But much of the material has had to be re-written to be up-to-date, and, of course, in accordance with the modern fashion, the book has much increased in size.

The chief changes are two in number. A brief chapter on The Design of Simple Experiments and a very long one on The Field Plot Test have been replaced by four chapters on Basic Experimental Designs, Factorial Experiments, Incomplete Block Experiments and The Treatment of Non-Orthogonal Data. The other change is that a final chapter on Miscellaneous Applications has been replaced by separate chapters on The Discriminant Function, Probit Analysis, and Quality Control and Sampling for Inspection and Verification.

The author has had much experience with the methodology of statistics, and, although not a mathematician, has taken pains to understand the mathematical models and proofs behind the currently accepted tests. He has a gift for exposition, and the reader who invests in this new edition may feel confident that he will bring himself up-to-date without many of the troubles attendant upon having to master the modern very-mathematical text with a slightly out-of-date mathematical equipment. J. WISHART.

Statistics for the Social Sciences. By T. G. CONNOLLY and W. SLUCKIN. Pp. vii, 154. 16s. 1953. (Cleaver-Hume Press)

The authors of this introductory text-book state that it is an attempt to help those students of Sociology, Economics, Psychology, or Medicine " who have no knowledge of mathematics beyond simple arithmetic ". Within the limits thus imposed by the mathematical attainments of their intended readers, the authors have written a careful and lucid exposition of descriptive statistics, correlation, and the testing of significance. The text is illustrated by numerical examples, though the usefulness of the book for class-work would have been enhanced by the inclusion of some exercises. Non-mathematical readers might also have appreciated the inclusion of more of the established graphical methods which would avoid tedious computation ; to the intended readers they would appear to be no more arbitrary than many of the mathematical results which they are asked to accept in this book.

There is no doubt that the book will be welcomed by students of the social sciences, but from this fact several questions arise. Is it possible to produce a sound understanding of statistical methods in students whose mathematical knowledge is so limited? Should it be necessary to write one book describing the basic ideas of statistics for social scientists, another for engineers, another for biologists, another for mathematicians, and so on? Is not the school the place for the teaching of an elementary subject so generally applicable?

Whatever the answers to these questions may be, this well-written book deserves a place in the school library. If statistics is taught in the Sixth Form this book will help to dispel the confusions of the weaker students ; if statistics is not taught it will remind would-be students of the social sciences that there is for them no escape from mathematics. B. C. BROOKES.

Volume and Integral. By W. W. ROGOSINSKI. Pp. ix, 160. 10s. 6d. 1952. University Mathematical Texts. (Oliver and Boyd)

This new and welcome addition to a well known collection provides a clear and, in view of its price and size, a surprisingly comprehensive introduction to the theories of measure and integration in Euclidean space R_n of n dimensions.

The first chapter should be particularly welcomed as it contains a more complete account of the theory of sets of points than is usually found in books of similar scope. The second chapter deals with Jordan content and the third with Lebesgue measure, the definition being based on that of inner as well as outer measure.

The next two chapters are concerned with integration, chapter four with Riemann integration and chapter five with Lebesgue integration. A novel feature is that the Riemann integral of a non-negative function $f(p)$ in R_n is defined as the content of an ordinate set in R_{n+1}, the usual assumption that $f(p)$ is bounded not being made. A similar definition is given for the Lebesgue integral, content being replaced by measure ; the extension to functions taking both signs is, in the case of either integral, made in the usual way. Chapter five concludes with a proof of Fubini's theorem and an account of the relation between the Riemann and Lebesgue integrals.

The sixth and last chapter contains a clear and illuminating account of the connection between the differentiation and integration of functions of one variable. Proofs of the second mean value theorem and of the formula for integration by substitution are also included.

Finally it should be mentioned that there are solutions to 35 examples, many of which are of considerable interest and importance. W. L. C. S.

Leçons de Géométrie Projective. By L. GODEAUX and B. ROZET. Pp. 278. 2800f. Second edition, 1952. (Edit. Sciences et Lettres, Lieget ; Masson, Paris)

The first edition of this book appeared in 1932, but was apparently not reviewed in the *Gazette*. It is a text-book on synthetic lines, the subject being treated in a pleasant and fairly elementary manner. The new feature of the present edition is an additional chapter of examples, with outlines of solutions. The book is attractively produced. J. A. T.

Ancient formal logic. By I. M. BOCHEŃSKI. Pp. vi, 122. Fl. 12. 1951. Studies in logic and the foundations of mathematics. (North-Holland Publishing Co., Amsterdam)

It is no adverse comment on this book, as it might well be on some, to say that it is another reminder that there is nothing new under the sun. The standpoint of the book is, in fact, both fresh and original—it is the discoveries about Greek logic which it brings to light and the revelation of the distance which the Greek logicians advanced along what have hitherto seemed to be characteristically modern lines, which prompt this denial.

Professor Bocheński blames the long delay in bringing the facts to light on the great influence exerted by the nineteenth century historian Carl Prantl whom he charges with " incredible misrepresentations ". The claims which the author makes for Greek logic are supported by more than 700 references to

original sources but much is still necessarily based on conjecture ; the author himself modestly sums up the position in the words " ancient formal logic is actually little more than a field for scientific research ".

In the introduction we are told that the book was written for logicians who are familiar with the technique and notation of Russell's *Principia*. Throughout the emphasis is on formalism and, aided of course by the use of modern symbolism, the similarity of the developments in formal logic from 350 to 200 B.C. to the logic of 1850–1950 A.D. is strikingly exhibited in hundreds of laws and formulae.

The ancient logicians whose contributions are examined in detail are Aristotle and the logicians of (what the author calls) the Stoic-Megaric school, Eubulides, Diodorus, Philo and Chrysippus. Aristotle's contributions to formal logic include the logic of classes (to which his famous syllogisms belong) and a theory of modal sentences. Aristotle also discovered a number of laws belonging to the logic of relations. In this connection Bocheński has an amusing story to tell about De Morgan, who is supposed to have said that the whole of Aristotle's logic would not prove that because a horse is an animal the head of a horse is the head of an animal. Russell and Whitehead defended Aristotle against this implied criticism by pointing to this lack as a merit in Aristotle's logic since the proposed inference is fallacious without an added existential premiss ; but Bocheński found in Aristotle's work (in the *Topica*) a *correct* formulation of a law by which the desired inference could be made.

To the Stoic-Megaric school is attributed the invention and axiomatisation of a logic of propositions, the invention of truth tables, a thorough discussion of the meaning of implication and a clear distinction between logical laws and metalogical rules of inference. None of the works of this school has been preserved and all the information which we have about them is contained in a few much later sources. Reference is made to a forthcoming book on Stoic logic by Dr. Benson Bates.

The book concludes with a list of 60 texts and a four-page bibliography followed by an index of some 240 Greek terms. R. L. GOODSTEIN.

Sentences undecidable in formalised arithmetic. An exposition of the theory of Kurt Gödel. By A. MOSTOWSKI. Pp. viii, 117. 12 fl. 1952. (North-Holland Publishing Company, Amsterdam)

In 1931, in a paper which will undoubtedly rank amongst the foremost contributions to the development of mathematical thought in the first half of this century, the Austrian mathematician Kurt Gödel established the impossibility of an intrinsic proof of freedom from contradiction of cardinal arithmetic. Gödel showed that every consistent formalised system of mathematics rich enough to contain recursive arithmetic necessarily contains undecidable statements, that is, statements which can neither be proved nor disproved by the proof resources of the system, and deduced that a consistent system cannot prove its consistency by means alone of the proof processes which the system admits. A consistency proof by methods which transcend the limits of the system itself is indeed possible, and such a proof was given by Gerhard Gentzen (later killed by the Nazis in Czecho-Slovakia) in 1936 ; Gentzen utilised transfinite induction (over the ordinals of the second class) to prove that certain reduction processes which his proof employs terminate in a finite number of steps. Gentzen's extrinsic proof of consistency has of course the defect—in relation to an intrinsic proof—that the system in which the proof is carried out is *not* the system whose consistency is proved, so that our confidence in the freedom from contradiction of arithmetic remains based on a belief in the freedom from contradiction of some wider system.

Quite apart from the striking results which he obtained in this paper (and in subsequent work) the methods which Gödel introduced have also led to widespread advances by other workers in the field, chiefly in the United States and Poland.

Professor Mostowski, who writes this account of Gödel's work, has himself made numerous contributions to its clarification and development. In the introduction Mostowski discusses informally the significance of undecidability and contrasts the notion of undecidability in a formal system with the concept of an essentially undecidable problem. A sentence may for instance be undecidable in some system S and yet decidable in some richer one S'; an essentially undecidable sentence (if there is one) would be undecidable from the very nature of mathematical reasoning. Mostowski stresses this point because Gödel's undecidable statement $(n)\,G(n)$ is such that each of $G(0)$, $G(1)$, $G(2)$, ... is true (and provable) so that from the standpoint of intuitive mathematics $(n)\,G(n)$ (or what we intend this to mean, namely $G(0)$ and $G(1)$ and $G(2)$ and so on) is both true and provable. Thus, for instance, if $G(n)$ says that $2n$ is a sum of two primes and if $G(0)$, $G(1)$, $G(2)$, and so on, are *all* provable, then every mathematician would consider that Goldbach's hypothesis had been proved even if $(n)\,G(n)$ is not formally derivable. The importance of Gödel's result is not that it throws doubt on the possibility of proving the famous hitherto unsolved problems (it is in fact completely silent on this point) but that it shows that mathematical induction and the rule of substituting for free variables do not suffice to express the intuitive notion of "any number" by a variable in a formal system (and Mostowski does not adequately bring out this essential point but only remarks that Gödel's undecidable statement shows that the formal system is not rich enough to express our intuitive mathematics).

In the first chapter Mostowski proves a number of purely arithmetical results for later use. These results serve to carry out one of the most important steps in Gödel's construction, the proof that a function defined by a primitive recursive definition admits an explicit definition in a formal system which is capable for expressing the idea "the least number which has a certain property". The essential ideas in this proof go back to Dedekind and von Neumann.

In Gödel's original paper he established a one-one correspondence between the elements, the formulae and the proof processes of a formal system and the class of natural numbers in such a way that the syntactical theorems of the system, such as "formula A follows by consequence rule x from the formulae B and C" or "proof number m proves formula number n", become equivalent to purely arithmetical formulae, thereby effecting a mapping of the syntax of the system on to the system itself. Mostowski short-circuits this process and takes the formal system itself to be a mere interpretation of a numerical system, so that, for example, certain classes of numerals are called the variables, other classes of numerals are called functions, and so on. This of course effects considerable economy in symbols.

In chapter III Mostowski builds up the arithmetic of this formal system S; this section could with advantage have been simplified by a wider use of the methods of recursive arithmetic, since the first chapter has already supplied the necessary justification. This is followed by an account of the semantics of S, that is, a formalisation of the notion of truth (as opposed to provable), and an analysis of the concepts of definability and, in particular, recursive definability of functions and relations. This brings us to chapter VI, which contains three proofs of Gödel's theorem, one due to Tarski, one following the lines of Gödel's original paper, and one based on Barkeley Rosser's fine generalisation of Gödel's theorem. These proofs are susceptible of a

variety of interpretation. The first, for instance, may be held to show that the class of true propositions is not definable in S. The chapter is exceedingly good and brings out more clearly than any other account known to the reviewer the necessary and sufficient conditions under which a Gödel type theorem can be established.

The book concludes with an appendix giving brief references to Gödel's work on the undecidability of the sentence expressing the consistency of S, and the construction of a hierarchy of systems each of which is capable of proving the consistency of its predecessor. Some account of Rosser's work on non-constructive logics and Turing's ordinal logics would have been helpful here.

Despite the very helpful introduction, this is not a suitable book for beginners in the field. The notes by Kleene and Rosser on Gödel's lectures (in English) at Princeton (though, alas, hard to obtain) provide a much simpler starting point. The chief merit of this volume—and it is a great merit—lies in its analysis of the notions of truth and definability and the application of this analysis in formulating minimum conditions for the existence of undecidable statements in a formal system. R. L. GOODSTEIN.

Contributions to the founding of the theory of transfinite numbers. By GEORG CANTOR. Translated with an introduction and notes by P. E. B. Jourdain. Pp. vii, 211. Paper \$1.25; cloth \$2.75. 1952. (Dover Publications, New York)

The re-issue of Jourdain's translation (prepared before the first world war) of two papers of Cantor's in the *Mathematische Annalen* for 1895 and 1897 is one of a new series being produced by the Dover Press; each volume is to be obtainable in two forms, one cloth bound, the other in paper wrappers to suit the impecunious student.

The articles translated are two in which Cantor presented a "final" form of the theory of transfinite cardinal and ordinal numbers which he had created some twenty years earlier. The ground covered by these articles is the arithmetic of transfinite numbers, including addition, multiplication and exponentiation of transfinite cardinals, and addition and multiplication of transfinite ordinals. Apart from the modifications in the definitions which Russell later introduced (by which a cardinal became a class of similar classes instead of "a concept abstracted from a class") and the subsequently discovered Zermelo's axiom of choice and Russell's theory of types, this original account of Cantor's would serve as well as any modern text to initiate a student in this exciting branch of mathematics.

In his introduction Jourdain gives a sketch of the development of the theory of functions up to Cantor's time and shows how Cantor arrived at the concept of a transfinite ordinal in his work on trigonometric series. The introduction serves also to summarise some of Cantor's earlier papers on transfinite numbers and the nature of mathematical entities in general.

The book concludes with some notes on developments in the theory of aggregates up to 1913. Amongst the results named are Russell's theory of types, the Schröder-Bernstein theorem, Hausdorff's theory of linear ordered aggregates and work by F. Riesz and L. E. J. Brouwer on multiply ordered aggregates.

In the forty years which have elapsed since the notes were written the theory of aggregates has continued to be a well-worked mine rich in new results. In one direction the development of the theory of measure by Borel, Lebesgue, Denjoy and Kolmogoroff has completely transformed the theory of

integration. The general theory of metric spaces is another fruitful branch of the theory of aggregates, and the immensely powerful theories of point-set topology also have their roots in Cantor's work (for example, in relation to aggregates of different dimensions) and Cantor was perhaps the first to stress the importance of continuous (1,1) correspondences. One of the most remarkable discoveries in this field (due in this form to Robinson) is that a solid sphere S may be divided into 5 mutually exclusive parts and these 5 parts reassembled by suitable *rigid motions* into 2 new spheres (one containing 3 of the parts and the other 2) *each sphere being of the same size as the original sphere S*.

During the first quarter of the present century the theory of infinite aggregates was the centre of a storm of controversy. Cantor believed that he could show that every aggregate can be well ordered, but in 1904 Zermelo and Schmidt showed that Cantor had taken for granted the existence of a class which contains at least one member of each of a given infinite set of classes, a result which appears to be incapable of proof and which has been postulated as the axiom of choice. Seven years earlier Burali-Forti pointed out that the conception of the series of all ordinals is self-contradictory since the ordinal of this series is both an ordinal and greater than every ordinal. This paradox was resolved by Russell's theory of types which separated objects, classes of objects, classes of classes of objects, and so on, into separate types and banned the formation of classes of mixed type. The theory of types obliged Russell, however, to introduce two further axioms, the axiom of infinity and the axiom of reducibility, to fill certain gaps which the theory of types itself created; the need for the second axiom, however, appears to have been obviated by simplifications which Ramsey made in the theory of types.

No progress has been made towards the elimination of the axiom of choice although a considerable number of equivalent propositions has been discovered. The axiom of choice is, for example, equivalent to each of the following propositions:

any infinite aggregate E may be (1,1) correlated to the class of all pairs (a,b) of elements a and b belonging to E;

in every closed family of sets there is at least one set which is not contained in any other set of the family;

for any two non-empty sets there exists a one-valued function which maps one set on the other.

Recently (1940) Kurt Gödel has shown that if set theory without the axiom of choice is free from contradiction then the introduction of the axiom does not introduce a contradiction into the theory of sets. A new codification of the theory of sets was published (in parts) during the last war and will shortly be issued in book form.

In 1936 Gentzen utilised transfinite induction for ordinals up to ϵ_0, the first ordinal to satisfy $\omega^{\epsilon_0} = \epsilon_0$, as a minimum deviation from finite proof processes to establish the freedom from contradiction of classical arithmetic. Subsequently ϵ_0 was shown to be the least initial ordinal for which transfinite induction was needed to prove that a descending sequence of ordinals is finite. Parts of Cantor's theory of transfinite ordinals have been formulated in recursive arithmetic independently of the axioms of the theory of sets, but the existence of transfinite cardinals beyond the cardinal of a simple sequence remains contrary to every shade of finitist opinion. R. L. GOODSTEIN.

Elemente der philosophie und der mathematik. By A. SPEISER. Pp. 115. 11.45 sw. fr. 1952. (Birkhäuser, Basel)

In the introduction Professor Speiser describes his book as finger exercises for beginners in philosophical and mathematical thought. Fundamental con-

cepts in philosophy and mathematics are subjected to the Hegelian dialectic to bring out their inner content. The seven stages of the dialectic are: thesis, antithesis, synthesis, synthesis with emphasis on being, with emphasis on nullity, and conflict between the latter leading to existence. Applied to the concept of infinity, for instance, we obtain in turn "Das unendliche. Es negiert das Endliche. Es is daher Negation der Negation und damit das wahrhafte Sein. Aber das Endliche selber geht über sich hinaus und negiert seine Negation. Das Unendliche ist daher die affirmative Bestimmung des Endlichen; Negation der Negation ist Affirmation. Das Endliche ist um Unendlichen verschwunded, denn die Unendlichkeit ist das Ansich sein des Endlichen".

The poverty of Hegel's own contribution to the foundations of mathematics was due, Speiser considers, to the lack of interest in mathematics in Germany at the time; France's Lagrange, Laplace, Legendre, Carnot, Dupin, Lamé, Monge, Poncelet, Lacroix and Cauchy were challenged in Germany only (but outstandingly) by Gauss. (One may wonder in this connection whether a country is a great military power because of the genius of its mathematicians or if it is rich in mathematicians because it is a great military power. Leibniz, of course, was both a great military strategist—Napoleon is said to have used his plan of campaign in the battle of the Nile—and a philosopher who made his own mathematics).

Speiser makes a number of interesting observations on the great mathematicians philosophising about their subject, and draws some striking parallels between mathematics and musical composition (the book in fact is written in fugue form) but the major part of the work is totally alien to English philosophical thought, and has no point of contact with present trends in the foundations of mathematics. R. L. GOODSTEIN.

Introduction to Metamathematics. By S. C. KLEENE. Pp. x, 550. Fl. 32.50. 1952. (Noordhoff, Groningen; North-Holland Publishing Co., Amsterdam)

This is the most important and substantial text-book of mathematical logic published since 1939, an exposition as brilliantly lucid as it is comprehensive. As Dr. Johnson might have said, if the reader but provide the understanding he will find here all the arguments he could wish for.

Works in this field appear under many titles. Whitehead and Russell (no doubt out of deference to Newton) called their three volumes *Principia Mathematica*. The two great volumes by David Hilbert and Paul Bernays were called *The foundations (grundlagen) of mathematics* and the more recent, and less ambitious work of Quine's simply *Mathematical logic*. Kleene's title is deliberately chosen to emphasise his interest in *constructive* methods since he defines metamathematics as the study of formal systems (logical, mathematical) by more or less informal, finitary techniques. These techniques are applied not only to classical systems admitting the *tertium non datur* but also to formalisations of intuitionistic logic and a comparison of the two logics is one of the features of the book.

Practically the whole of the book is devoted to the metamathematical study of elementary number theory and, apart from some purely introductory remarks, there is no reference to analysis, type theory or set theory. The gap left by the omission of set theory will however shortly be filled by the forthcoming *Axiomatisation of Set Theory* by Bernays, Fraenkel and Borgers. Analysis must await its turn.

The set of axioms for the basic propositional calculus (due to Gentzen) are chosen to bring classical and intuitionistic logic as close together as possible,

the systems differing only in a single axiom with "$\neg\neg A = A$" (elimination of double negation) in the classical case replaced by "$\neg A = (A = B)$" in the other. The operations of the calculus are described by means of Gentzen *sequents*; so that the fact that a formula E is derivable from a sequence of formulae Γ by means of the axioms and derivation rules of the calculus is denoted by $\Gamma \vdash E$. The deduction theorem (which serves for the elimination of hypotheses) is proved in the form "$\Gamma \vdash A = B$ follows from Γ, $A \vdash B$" and a series of deduction rules for the introduction and elimination of the logical constants is established along the lines of Gentzen's 1934–5 paper. Truth tables are introduced to provide a set of valuations for formulae of the propositional calculus by which the completeness and freedom from contradiction of the set of axioms is proved. A rather surprising omission from this section is the lack of a proof of the independence of the axioms. Truth tables provide a decision procedure for the classical propositional calculus, but according to a result of Gödel's the intuitionistic calculus cannot be treated on a truth table basis for any finite n, though a table of $\aleph_0$ values has been shown to suffice.

The system of elementary number theory which Kleene studies is based on mathematical induction, the Peano axioms $a' = b' = a = b$, $\neg a' = 0$, the transitive law for equality, the single-valuedness of the stroke function and the recursive equations for addition and multiplication. Reference is made to Presburger's theorem that the omission of the axioms for multiplication renders the resulting system demonstrably consistent and complete, but no proof is given (and in fact the omission of the axioms for *either* multiplication or addition suffice). Presburger's result is contrasted with the remarkable incompleteness theorem which Gödel obtained in the following year (1931). Gödel proved that if elementary number theory is simply consistent (that is, for no formula A are both A and $\neg A$ provable) there is a formula $\forall b \neg A(p, b)$ which is unprovable and if the system is ω-consistent (that is, for all A $A(0)$, $A(1)$, $A(2)$, ... , and $\neg \forall x A(x)$ are not all provable) then also the negation of this formula is not provable, so that there is a formula which is neither provable nor refutable. Kleene gives both this original form of Gödel's theorem and also Rosser's extension in which the requirement of ω-consistency is replaced by simple consistency. From his proof of incompleteness Gödel deduced that if elementary number theory is consistent then no proof of consistency can be formalised in the system itself. Kleene does not give the details of this proof (which are not to be found in the original Gödel paper) but refers the reader to the second volume of the Hilbert-Bernays *Grundlagen*.

Kleene's first account of Gödel's theorem takes for granted a number of results proved in later chapters. Presumably the object of this is to present the result in the first instance with a minimum of technical details; certainly the device makes the result accessible to many who would find the relevant parts of recursive function theory a little tedious. In completing the proof Kleene follows Gödel in utilising the *numeralwise expressibility* of recursive functions, which means that to any primitive recursive function $\phi(x_1, x_2, \ldots x_p)$ corresponds a formula $P(x_1, x_2, \ldots, x_p, y)$ of the calculus such that for any set of integers $\xi_1, \xi_2, \ldots, \xi_p, \eta$ satisfying $\phi(\xi_1, \xi_2, \ldots, \xi_p) = \eta$, the formula

$$P(m_1, m_2, \ldots, m_p, n)$$

is provable, where $m_1, m_2, \ldots, m_p, n$ represent the integers $\xi_1, \xi_2, \ldots, \xi_p, \eta$ in the calculus. The proof of numeralwise expressibility is almost as long and difficult as the proof (in the *Grundlagen*) of the wider property that to any primitive recursive function corresponds in the calculus (suitably fortified by a minimal operator) a functional term for which the corresponding recursion

equations are *provable* in the calculus. For instance, if $f(n)$ is the term and $f(n')=a(n, f(n))$ its recursive equation, then this equation is provable with a variable n, but under the hypothesis of numeralwise expressibility the most that is known is that any equation which results from the recursion equation by substituting a definite numeral for the variable n, is provable. Apart from the assurance it gives that the incompleteness of the system is not a consequence of the introduction of the minimal operator, it is hard to see what advantage there is in introducing numeralwise expressibility when the deeper result of the *Grunedlagen* is available.

After a brief section on various extensions of primitive recursion Kleene introduces the property of general recursion, in an original formulation of the author's based on a suggestion of Herbrand's. A function $\phi(x_1, x_2, \dots, x_n)$ is said to be general recursive (though the term is an unfortunate one) if there is a finite set of equations from which for any set of numerals $\xi_1, \xi_2, \dots, \xi_n$ an equation of the form $\phi(\xi_1, \xi_2, \dots, \xi_n)=\eta$ is derivable for a certain unique η, using only the operations of substituting numerals for variables, and equals for equals. Primitive recursive functions are general recursive, as is also every extension of primitive recursion, such as multiple recursion. Recently Ackermann introduced the important notion of transfinite recursion and Peter showed that the class of general recursive functions transcends the class of multiply recursive functions by constructing a function of transfinite ordinal ω which is general recursive but not n-ply recursive for any n. A detailed study of transfinite recursion promises to throw fresh light on the boundary between constructive and non-constructive processes.

Kleene accepts Church's thesis that every *effectively calculable* function (effectively decidable predicate) is general recursive. The evidence for this thesis consists primarily in the fact that no one has so far discovered an example which contradicts it, and also in the corroboratory circumstances that no alternative definition (Turing's *computability*, Church's *λ-definability*, Post's *normal systems*) which has been proposed leads to a wider class of functions than the general recursive. There can be no doubt of the importance of the concept of the general recursive function. This importance derives from the following theorems (due to Kleene):

1. If $\mu y R(y)$ denotes the least y such that $R(y)$ holds, then every general recursive $\phi(x_1, x_2, \dots, x_n)$ is expressible in the form $\mu y R(x_1, x_2, \dots, x_n, y)$, where R is a primitive recursive relation and $EyR(x_1, x_2, \dots, x_n, y)$ holds.

2. There is a particular primitive recursive relation (for each n)

$$T_n(x_1, x_2, \dots, x_n, y)$$

such that for any general recursive predicate $R(x_1, x_2, \dots, x_n, y)$ numbers f and g can be found for which

$$(Ey)\, R(x_1, x_2, \dots, x_n, y) \equiv (Ey)\, T_n(f, x_1, \dots, x_n, y),$$
$$(y)\, R(x_1, x_2, \dots, x_n, y) \equiv \quad (y)\, T_n(g, x_1, \dots, x_n, y).$$

2.1. The class of relations expressible by means of a succession of quantifiers prefixed to a relation R is the same whether R is restricted to primitive recursive relations or general recursive relations are admitted.

3. For a given formal system S the predicate "$A(x)$ is provable in S" is expressible in the form $(Ey)\, R(x, y)$, where R is general recursive.

In spite of its importance, however, it is difficult to accept general recursive as a characterisation of effective calculability. Kleene himself says that a general recursive $\mu y R(x_1, x_2, \dots, x_n, y)$ is only as constructive as the existence relation $Ey\, R(x_1, x_2, \dots, x_n, y)$ from which it is derived. This, of course, leads to a vicious circle, for what criterion of constructiveness for the proof of the existence relation $Ey\, R(y)$ are we offered, if not the general recursiveness of the y which satisfies the relation? It appears, therefore, that we must look

elsewhere for the criterion, and it seems possible that it will be found in the series of transfinite recursions (of ordinals less than ϵ for which the finiteness of a decreasing sequence is provable without recourse to transfinite induction).

Kleene obtains a generalisation of Gödel's theorem to the effect that no formal system can be complete for the purpose of proving all the true propositions (and no others) taken as the value of a certain predicate

$$(y)\bar{T}_1(x, x, y);$$

this shows that the incompleteness which Gödel discovered is not just an accidental feature of a particular formal system. A similar generalisation of Rosser's form of Gödel's theorem is also given. This part of the book concludes with an excellent account of Turing's concept of a machine and a very simple proof of the equivalence of the notions of general recursiveness and machine computability.

Gödel's completeness theorem for the predicate calculus is proved in the form : If for a predicate formula F, $\neg F$ is unprovable in the calculus then F is satisfiable in the domain of natural numbers. The proof which Kleene gives is novel and attractive and the form in which the result is obtained immediately yields the famous Löwenheim theorem that any predicate which is satisfiable in a non-empty domain is satisfiable in the domain of natural numbers. Löwenheim's theorem leads to the paradox that a formalisation of set theory in the predicate calculus admits a denumerable model ; that is, the primitive notions can be interpreted so that there are only enumerably many sets and all the axioms are true even though a theorem of the theory asserts that there is a non-enumerable infinity of sets. Kleene explains the paradox by interpreting it as an incompleteness theorem which asserts that the class of pairs which correlates the sets of the theory to the natural numbers is not definable within the formalisation of the theory. As Skolem observed, the paradox forces us to admit either that the concept of a non-enumerable set eludes characterisation in a formal system of the type under consideration or that we must regard the concept as a relative one, so that a set which is non-enumerable in one axiomatisation may become enumerable in another.

A long and detailed account of Gentzen's formal system of sequents, proving the eliminability of the particular derivation schemata called " cut " and " mix ", leads up to a proof of the consistency of number theory without induction, but Gentzen's proof of the consistency of number theory with an unrestricted induction postulate (utilising transfinite induction to escape the consequences of Gödel's theorem) is given only in most disappointingly unhelpful outline. The concluding chapters of the book are devoted mainly to intuitionistic systems. It is shown that classical number theory is simply consistent if intuitionistic number theory is, by means of a theorem of Gödel's to the effect that if E follows classically from a sequence of predicates Γ then E^0 follows intuitionistically from the sequence Γ^0 where E^0 comes from E (and each predicate of Γ^0 from the corresponding predicate of Γ) by substituting for each part of E the operation shown in the second line for that in the first :

$$\begin{array}{llllll} A \supset B & A \,\&\, B & A \vee B & \neg A & \forall x A(x) & \exists x A(x) \\ A \supset B & A \,\&\, B & \neg(\neg A \vee \neg B) & \neg A & \forall x A(x) & \neg\forall x \neg A(x). \end{array}$$

Concerning this theorem Gödel remarks that it shows that intuitionistic arithmetic is only apparently narrower than the classical since the intuitionistic includes the entire classical number theory with a somewhat differing interpretation. To underline the importance of this last qualification Kleene

formulates an interpretation of intuitionistic number theory under which the formula

$$\neg\forall x\,(A(x) \vee \neg A(x)),$$

which is the negation of a classically provable formula, is true intuitionistically.

Kleene has taken great pains to include references to recent discoveries, and there are in fact references to papers which have not yet appeared. Some omissions which perhaps call for mention relate mainly to constructive systems other than the intuitionistic. These include papers of F. B. Fitch in the *Journal of Symbolic Logic* on extensions of basic logic, a paper by J. R. Myhill on a complete theory without negation, and another by M. H. Löb (forthcoming) in the same journal. There is also an extension of Gödel's theorem (in the reviewer's *Constructive Formalism*) which constructs a *numerical* term λ of the predicate calculus such that for any natural number $\mathbf{n}$, the equation $\lambda = \mathbf{n}$ is unprovable. The most surprising omission of all, however, is the lack of any reference to formalisations of metamathematics such as that given in Quine's *Mathematical Logic*.

Kleene's book is noteworthy for the impressive fidelity of its detail, its balance, its carefully chosen notations and its mastery of technique, but there is nothing in it of philosophy or speculation. On the nature of the cardinal numbers, of variables, of functions, or mathematical induction, the book is silent; in this Kleene is a faithful representative of the new school of mathematical logic which is contented with its supereminence in technology and leaves to others the discovery of the underlying epistemology.

R. L. GOODSTEIN.

Calculus, A Modern Approach. By K. MENGER. Pp. xxv, 255. Mimeographed edition. $4.50. 1952. (Illinois Institute of Technology)

Professor Menger likens our teaching of calculus to a man's coat: it is "full of buttons without any possible use". In this book he removes all the superfluous "buttons", and perhaps some of the necessary ones as well, and he calls the result "streamlined calculus".

The aim of the author is to get rid of redundant symbols and to build up an unambiguous notation. Thus, since

$$\int_a^b f(x)\,dx, \quad \int_a^b f(t)\,dt, \quad \int_a^b f(u)\,du,$$

all have the same value, he writes $\int_a^b f$. Trouble at once arises if one tries to drop the x and dx in

$$\int_a^b x\,dx, \quad \int_a^b x^2\,dx, \quad \text{etc.}$$

In order to overcome this difficulty, Professor Menger introduces the symbol I for the line $y = x$ and I^n for the curve $y = x^n$. Then $\int_a^b x^n\,dx$ is written $\int_a^b I^n$.

In the statement

$$\frac{d}{dx}\sin x = \cos x,$$

the use of x is ambiguous because it is possible to give x on the right-hand side

any value—say $\frac{1}{2}\pi$—but not on the left-hand side. Professor Menger therefore writes

$$\mathfrak{D}\sin = \cos \text{ (in general)} \quad \text{and} \quad \mathfrak{D}\sin \tfrac{1}{2}\pi = \cos \tfrac{1}{2}\pi.$$

This example brings out one of the difficulties of the new notation. Does $\mathfrak{D}\sin \frac{1}{2}\pi$ mean $\frac{d}{dx}(\sin \frac{1}{2}\pi)$ or $\left[\frac{d}{dx}\sin x\right]_{x=\frac{1}{2}\pi}$? Definite rules of association are laid down which decide this point.

With the I-notation the y-axis is dispensed with and the author talks of the parabola I^2, the half parabola $I^{\frac{1}{2}}$ and the hyperbola I^{-1}, only the x-axis being drawn and graduated. One grave defect of this notation is that although I^2 and $I^{\frac{1}{2}}$ are both parabolas, they have no similarity of form in the way that $y=x^2$ and $x=y^2$ have. The curves of the trigonometrical and transcendental functions are denoted by sin (for $y=\sin x$), cos (for $y=\cos x$), etc., and $\exp_a$ (for $y=a^x$), $\log_a$ (for $y=\log_a x$). The value of the ordinate of f at (say) $x=2$ is denoted by $f2$ (without brackets) and in the same way

$$I^n 2 = 2^n, \quad \exp_a 2 = 2^a.$$

Because drill in the use of the symbol I is necessary, Professor Menger first develops a miniature calculus for straight lines. For this no limiting processes are needed. The criticism here is that the work involved seems so pointless. Everything can be done so much more easily by trigonometry and by simple mensuration that the work of the initial chapter is likely to breed a distaste for the subject. In Chap. II the work of Chap. I is extended to polygons and in Chap. III it is used for finding approximate areas under curves and approximate gradients. It is not until Chap. VI (p. 100), after a discussion of functions (Chap. IV) and limits (Chap. V) that one reaches the calculus proper.

Here one meets the definition of the derivative of $g(x)$ in the form

$$\mathfrak{D}gx = \lim_{\bar{I}-x} \frac{g-gx}{\bar{I}-x} \left(\text{for } \frac{d}{dx}g(x) = \lim_{x' \to x} \frac{g(x')-g(x)}{x'-x}\right).$$

The reverse process to derivation is called anti-derivation, so that, for example, $I^n/n+C$ is an anti-derivative of I^{n-1} $(n \neq 0)$ for all C.

The term " integral " is at first reserved for definite integrals and on p. 107 the following definition of $\int_a^b f$ is given in terms of areas :

" The area under the portion from a to b of a simple curve f is the limit of the areas under the regular step lines with n steps [i.e. *each of width* $(b-a)/n$] whose mid-points lie on the curve, as n approaches infinite." A " simple curve " is defined in Chap. I as one which intersects any line perpendicular to the x-axis in at most one point. Now, if

$$f(x) = \begin{cases} 1 & (x \text{ irrational}) \\ 0 & (x \text{ rational}). \end{cases}$$

then, according to this definition the graph of $f(x)$ is a simple curve. The mid-points of the n steps will all correspond to rational or irrational values of x according as a and b are both rational or one is rational and one irrational, and so

$$\int_a^b f = \begin{cases} 0 & (a \text{ and } b \text{ rational}) \\ b-a & (a \text{ rational}, b \text{ irrational}). \end{cases}$$

On p. 109 qualifying conditions are given which rule my example out, but why give a wrong definition in the first place? I suspect that it is in order to avoid an unwelcome dx.

The student who has worked right through the book should be competent to deal with certain problems: gradients of curves, areas under curves, the first mean-value theorem, Taylor's theorem (with remainder), tangent planes to surfaces, and volumes under surfaces. But what about a simple volume of revolution or a centre of gravity? With dx banished, it is not easy to see how one is going to set about such problems.

Large sections of the book are concerned with ideas about functions and variables and with general functional notation. I think that the latter is introduced far too early if the book is really meant for beginners. (See the report on *The Teaching of Calculus in Schools*, § 4.6). Some of the passages dealing with scientific variables and descriptive functions are highly controversial. Owing to criticisms already received, the author is going to modify the treatment in the next edition.

The main value of Professor Menger's book will, I think, lie in the interest which it will arouse in the presentation of the subject and not in its use as a text-book. Parts of it are very good and could be studied by teachers with profit, but I doubt whether many will venture to put it into the hands of their pupils. I. W. BUSBRIDGE.

Aufgabensammlung zu den gewöhnlichen und partiellen Differentialgleichungen. By G. HOHEISEL. 2nd edition. Pp. 124. DM. 2.50. 1952. Sammlung Göschen, 1059. (W. de Gruyter, Berlin)

The first edition of this useful compilation was reviewed in the *Gazette*, Vol. XVII, p. 142. Some improvements in arrangement and one or two small changes in content have been made in the new edition. Basic points of theory are briefly described, and illustrated by worked examples, and there is an excellent collection of representative exercises, some attached to the relevant sections, others in a "mixed bag", complete with answers and, in many instances, helpful hints. Like other volumes in the Göschen series, splendid value for the money. T. A. A. B.

Calculus of Variations with Applications to Physics and Engineering. By R. WEINSTOCK. Pp. x, 326. 55s. 6d. 1952. International Series in Pure and Applied Mathematics. (McGraw-Hill)

A much-loved Cambridge lecturer in analysis once remarked to his class; "Gentlemen, there are many excellent treatises from which you may study the theory of the calculus of variations; but if you wish to learn how to apply this calculus to special problems, I can only recommend you to ascend to the upper shelves of the University Library and blow the dust off the pages of Todhunter's *Integral Calculus*." This situation has been materially improved by the recent book by Dr. Fox, and by the present volume.

Professor Weinstock regards the calculus of variations as a tool to be used, not as a research field for the pure mathematician. Thus, while striving for reasonable rigour, he is concerned with necessary rather than with sufficient conditions, second variations are not studied, and there is little of the modern theory stemming from Jacobi and Weierstrass; in fact, the analysis might be not unfairly described as Eulerian with a dash of epsilonology. This, of course, is pitching the standard just about right for the user; he does not want to work through hundreds of pages on existence theorems before tackling a simple problem. In this book, after a brief preliminary survey of background material, we go at once to problems—the brachistochrone, isoperimetrical problems, geometrical optics, dynamics. With the extension to

more than one variable, and applications to vibrating strings and membranes, elasticity, quantum mechanics and electrostatics, manipulation naturally becomes more complicated but by working as a rule from the problem to the method the exposition remains comparatively simple and easy to follow. There is a danger that the unsophisticated reader may be led to believe that the pure mathematicians have been making a fuss about nothing, a danger not likely to be wholly countered by reference to a brief bibliography of some fifteen titles (containing Bliss, Bolza, and Courant-Hilbert, but not Carathéodory), and perhaps a few more warnings in the text would have been advisable. Nevertheless, for its own specific purpose, the book is well devised and clearly written, and should prove most helpful, the first part being particularly easy to read. A welcome feature is the generous supply of exercises for the student, about 150 in all.

Mémorial des Sciences mathématiques.

Sur quelques propriétés des valeurs caractéristiques des matrices carrées. By M. PARODI. Pp. 64. 800 fr. 1952. CXVIII.

Vorticity and the thermodynamic state in a gas flow. By C. TRUESDELL. Pp. 51. 900 fr. 1952. CXIX.

Complexes linéaires. Faisceaux de complexes linéaires. Suites et cycles de complexes linéaires conjugués. By A. CHARRUEAU. Pp. 83. 900 fr. 1952. CXX.

Sur les applications de la notion de moment d'inertie en géométrie. By D. WOLKOWITSCH. Pp. 41. 450 fr. 1952. CXXI.

Les transformations birationnelles du plan. By L. GODEAUX. Pp. 70. 900 fr. 1953. CXXII.

Mémorial des sciences physiques.

Physique des guides d'ondes électromagnétiques. By T. KAHAN. Pp. 100. 1000 fr. 1952. LII.

Condensation et adsorption des molécules sur une surface en atmosphère raréfiée. By F. M. DEVIENNE. Pp. 87. 1000 fr. 1952. LIII.

Propriétés optiques des lames minces solides. Pp. 84. 1000 fr.
Applications optiques des lames minces solides. Pp. 54. 700 fr. By M. P. ROUARD. 1952. LIV, LV. (Gauthier-Villars)

These two admirable series of monographs are so well-known and so widely recognized as authoritative expositions that to give a general recommendation is certainly superfluous. But while the present list of titles does not preclude a more generous notice of some or all in later issues of the *Gazette*, we take this opportunity of bringing the appearance of these, the most recent additions to the Gauthier-Villars list, to the immediate notice of our readers. It need hardly be said that the format is at the uniform level of excellence which we expect and receive from Gauthier-Villars.

Algebra, I, II. By G. CHRYSTAL. 6th edition, reprinted. Pp. xxiv, 571; xxiii, 616. $8. 1952. (Chelsea Co., New York)

Frequently in recent issues of the *Gazette* reviewers have commended the admirable series of reprints produced by the Chelsea Company. But in this instance we may wonder whether the expenditure of labour and paper should have gone on the reproduction of a museum piece. "Chrystal" was a great book, and it played its part in the late nineteenth century effort to replace the solving of Tripos problems by creative mathematics. But today no one would regard Chrystal's exposition as suitable for the novice, or his climate of thought as an inspiration for the Sixth Form.

For some of us, however, sentiment will conquer logic, and the book wherein we first encountered Bernoulli's numbers, continued fractions, the Riemann surface, will be sure of a welcome and a nostalgic turning of its pages before it is sent to accumulate dust on the top shelf. T. A. A. B.

Degree Mathematics; Formulae and Examples. By J. BLAKEY. Pp. 137 7s. 6d. 1952. (Cleaver-Hume)

This work consists of a statement (without proof) of all the formulae required for the General and Engineering degree papers of London University. They are divided into sections, each of which ends with a list of problems solvable by means of the preceding formulae. All of these problems have been taken from past papers set in the above examinations and altogether are about 400 in number.

The book will obviously prove useful to those students who rely only upon lecture notes supplemented by practice in the working out of problems.

P. M. H.

Anfangswertprobleme bei Partiellen Differentialgleichungen. By R. SAUER. Pp. xiv, 229. DM 26; geb. DM 29. 1952. Grundlehren der mathematischen Wissenschaften, 62. (Springer, Berlin)

This book is concerned with the behaviour of hyperbolic differential equations with given initial conditions. Such equations and systems of such equations occur in many physical problems and, comparatively recently, fresh impetus has been added to their study by the growing importance of supersonic flow problems. The book consists of two introductory chapters, a chapter on systems of quasi-linear first-order differential equations and the general second-order equation in two variables, and a chapter on the quasi-linear second-order differential equation in more than two variables.

Since the book is intended for applied mathematicians, the reviewer (a pure mathematician) hesitates to criticize its standard of accuracy, clarity and correctness. The author evidently believes, with the late Oliver Heaviside, that "definitions make themselves". Of the few definitions given, that of "simple connection" on p. 17 is incorrect and this is a particularly unfortunate mistake since it is one into which a newcomer to the subject is likely to fall. There are also errors amongst the footnote references which are too numerous to be detailed in a short review.

The book contains a comprehensive account of the subject matter from the point of view of applications but it appears to have been prepared with undue haste and with insufficient attention to detail. H. G. EGGLESTON.

MATHEMATICAL FILM

Plucked Strings. A mathematical film. 16 mm., silent, 5 min. By T. J. FLETCHER. (Sir John Cass College)

Although much has been spoken and written about the place of the film in the teaching of mathematics, this is the first film of real mathematical content produced in this country since the work of Fairthorne and Salt before the war. It uses the same animated diagram technique, but is simpler to follow, and is closer to the needs of teacher and student. The first sequence shows the vibrations of a taut string which is drawn aside at a point of trisection and released from rest. The second sequence shows the vibrations produced when the second point of trisection is held.

The drawings and photography are well done, and should be helpful to teachers. The film is available in a single length, or in two loops. Mr. Fletcher is also producing a film on the Simson Line, which promises to be of some interest. I. R. V.

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THOMAS ARTHUR ALAN BROADBENT, M.A.

President, April, 1953—January, 1954